Polynomially Bounded Logic Programs with Function Symbols: 
A New Decidable Class

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Abstract
A logic program with function symbols is called finitely 
ground if there is a finite propositional logic program whose 
stable models are exactly the same as the stable models of 
this program. Finite groundability is an important property 
for logic programs with function symbols because it makes 
feasible to compute such program’s stable models using tradi-
tional ASP solvers. In this paper, we introduce a new decid-
able class of finitely ground programs called POLY-bounded 
programs, which, to the best of our knowledge, strictly con-
tains all decidable classes of finitely ground programs discov-
ered so far in the literature. We also study the related com-
plexity property for this class of programs. We prove that 
deciding whether a program is POLY-bounded is EXPTIME-
complete.

Introduction
A logic program with function symbols II is called finitely 
ground if there is a finite propositional logic program II’ 
such that II and II’ have the same collection of stable mod-
els. Therefore, a finitely ground logic program will have a 
finite number of stable models and each stable model is of a 
finite size. Finite groundability is an important property 
for programs with function symbols because this makes feasi-
ble to compute such programs’ stable models using tradi-
tional ASP solvers (Calimeri et al. 2008; Basilice, Bonatti,
and Criscuolo 2009a; Alviano, Faber, and Leone 2010).

Unfortunately, in general, checking whether a program is 
finitely ground is undecidable. In recent years, several 
decidable classes of finitely ground programs have been 
discovered under the stable model semantics (Gelfond and 
Lifschitz 1988); ω-restricted programs (Syrjänen 2001), λ-
restricted programs (Gebser, Schaub, and Thiele 2007), fi-
nite domain programs (Calimeri et al. 2008), argument-
restricted programs (Lierler and Lifschitz 2009), safe pro-
grams (Greco, Spezzano, and Trubitsyna 2012), Γ-acyclic 
programs (Greco, Spezzano, and Trubitsyna 2012), and 
what we refer as GMT-bounded programs, which has been 
shown to be a proper superclass of all other previous classes 
(Greco, Molinaro, and Trubitsyna 2013). More recently, an-
other decidable class of finitely ground programs called size-
restricted programs was further introduced in (Calautti et 
al. 2015), in which it was shown that although this class 
does not contain the argument-restricted and GMT-bounded 
classes, the underlying technique may be combined with 
other approaches and eventually to identify more finitely 
ground programs (see next section of an overview on this).

However, it comes to our attention that some simple pro-
grams which are finitely ground but do not belong to ei-
ther of the class of GMT-bounded programs or size-restricted 
programs. Let us consider a scenario of an online photo 
gallery, where each paid member can view any image in 
the gallery, but a restriction is imposed for guest members: 
Although a guest member is allowed to view the gallery im-
ages, he/she can only view no more than two images in either 
large or thumbnail form each time. This may be expressed 
by the following two rules:

\[ r_1 : \text{viewLarge}(X, Y) \lor \text{viewThumbnail}(\text{next}(X), Y) \]
\[ r_2 : \bot \leftarrow \text{viewThumbnail}(\text{next}(X), Y). \]

Let \( \Pi_1 = \{r_1, r_2\} \). Then it follows that program \( \Pi_1 \) is 
finitely ground because program \( \Pi_1 \cup D \) will only have finite 
stable models for any given input database \( D \).

Surprisingly, \( \Pi_1 \) is not bounded under the definition of 
(Greco, Molinaro, and Trubitsyna 2013), nor is size-
restricted as specified in (Calautti et al. 2015). Motivated 
from this example, in this paper, we propose yet another 
decidable class of logic programs with function symbols called 
POLY-bounded programs, which strictly contains both GMT-
bounded and size restricted programs. The reason we are 
able to obtain such a strict class is that we give explicit treat-
ments to “disjunctions”, “negations” and “constraints” in the 
underlying programs.

The rest of the paper is organized as follows: Section 
2 presents necessary terminologies and background knowl-
edge we will need throughout the paper. Section 3 defines 
a fixpoint upper bound of all stable models for a given pro-
gram with function symbols. Based on this upper bound def-
inition, Section 4 then specifies a new decidable class of 
programs called polynomially bounded programs, and proves 
its main properties. Section 5 studies the complexity prop-
erty of this new decidable class of programs, and finally Sec-
tion 6 concludes the paper with some remarks.
Preliminaries

A disjunctive logic program (or simply called program) $\Pi$ is a finite set of rules $r$ of the form:

$$A_1 \lor \ldots \lor A_k \leftarrow B_1, \ldots, B_l,$$

where $A_1, B_1, \ldots, A_k$ are atoms for all $1 \leq i \leq h$, $1 \leq j \leq l$ and $1 \leq h \leq m$. We denote by $Hd(r)$, $Pos(r)$, and $Neg(r)$ the sets $\{A_1, \ldots, A_k\}$, $\{B_1, \ldots, B_l\}$, and $\{C_1, \ldots, C_m\}$, which are called $r$’s head, positive body, and negative body, respectively. Sometimes for convenience, we may simply denote rule $r$ by $Hd(r) \leftarrow Pos(r) \land \neg Neg(r)$. When $k \leq 1$ for all $r \in \Pi$, $\Pi$ is called a normal program; if for a rule $r$, $k = 0$, $r$ is called a constraint and we denote its head by $\bot$ and when $Pos(r) \cup Neg(r) = \emptyset$, $r$ is called a fact.

Given a predicate $p$ of arity $n$, the $i$-th argument of $p$ is an expression of the form $p[i]$. Throughout this paper, we denote by $\text{arity}(p)$ as $p$'s arity, and refer to an argument as $p[i]$ for $1 \leq i \leq \text{arity}(p)$. We denote by $\text{arg}(\Pi)$ as the set of all arguments of $\Pi$, $\text{pred}(\Pi)$ as the set of all predicative symbols in $\Pi$, and $\text{atoms}(\Pi)$ as the set of all atoms mentioned in $\Pi$.

A program $\Pi$ is range restricted if for each rule, the variables occurring in the head or in the negative body also appears in the positive body of that rule. As in (Greco, Molinaro, and Trubitsyna 2013), in this paper, we assume that all programs are range restricted. Given an atom $\alpha$, we denote by $\text{var}(\alpha)$ and $\text{const}(\alpha)$ as the sets of variables and constants occurring in $\alpha$, respectively. Moreover, we naturally extend this notion to a program $\Pi$ (to a set of atoms $A$) such that $\text{var}(\Pi)$, $\text{const}(\Pi)$ and $\text{atoms}(\Pi)$, resp. denote the set of variables, constants and atoms occurring in $\Pi$ ($A$, resp.), respectively. For convenience, we further denote by $\text{var}\text{const}(\Pi)$ ($\text{var\text{const}}(A)$) as the union $\text{var}(\Pi) \cup \text{const}(\Pi)$ ($\text{var}(A) \cup \text{const}(A)$), resp.

Given two sets of atoms $A_1$ and $A_2$, we say that $A_1$ is embeddable into $A_2$, denoted as $A_1 \ll A_2$, if there exists a mapping $\theta : \text{var}(A_1) \rightarrow \text{varconst}(A_2)$ such that for each atom $a_1 \in A_1$, there exists some atom $a_2 \in A_2$ such that $a_1 \theta = a_2$.

Now for a given program $\Pi$, by $\text{hu}(\Pi)$ and $\text{hb}(\Pi)$, we denote $\Pi$’s Herbrand universe and Herbrand base, respectively. Specifically, $\text{hu}(\Pi)$ is the set of all ground terms that can be built using the constants and function symbols in $\Pi$ (if $\Pi$ does not contain any constant, we introduce a constant in $\Pi$’s domain), while $\text{hb}(\Pi)$ is the set of all atoms that can be built from terms in $\text{hu}(\Pi)$ and predicate symbols of $\Pi$. Clearly, both $\text{hu}(\Pi)$ and $\text{hb}(\Pi)$ can be infinite. We say that a set of atoms $I$ is an interpretation of $\Pi$ iff $I \subseteq \text{hb}(\Pi)$. A rule $r'$ is a ground instance of $r \in \Pi$ if $r'$ is obtained from $r$ by substituting each variable in $r$ by some ground term from $\text{hu}(\Pi)$. We use $\text{ground}(r)$ to denote all ground instances of $r$, and $\text{ground}(\Pi) = \bigcup_{r \in \Pi} \text{ground}(r)$ as the grounding of the program $\Pi$, which could be infinite. Given an interpretation $I$ and a ground rule $r'$, we say that $I$ satisfies $r'$, denoted as $I \models r'$, iff $I \cap \text{hd}(r') \neq \emptyset$ whenever $I \subseteq Pos(r)$ and $I \cap Neg(r) = \emptyset$ holds. Then we say that $I$ is a model of a ground program $\Pi'$, denoted as $I \models \Pi'$, iff $I \models r'$ for all $r' \in \Pi'$.

Given an interpretation $I \subseteq \text{hb}(\Pi)$ and the grounding $\Pi' = \text{ground}(\Pi)$ of $\Pi$, we denote by $(\Pi')'$ as the reduced (or reduct) of the (ground) program $\Pi'$ such that it is denoted as the set of rules $\{\text{hd}(r) \leftarrow \text{pos}(r) \mid r \in \Pi' \}$ and $\text{neg}(r) \cap I = \emptyset$. Then we say that $I$ is an stable model of $\Pi$ iff $I$ is the minimal set that satisfies all the rules in $((\Pi')')'$ (Gelfond and Lifschitz 1988; 1991).

A program $\Pi$ is in functional normal form if for each $p(t_1, \ldots, t_k) \in \text{atoms}(\Pi)$, $\text{dep}(t_i) \leq 1$ for all $1 \leq i \leq k$, where $\text{dep}(t_i)$ denotes the greatest term depth of a complex term in $t_i$. It is obvious that for a given program $\Pi$, by introducing new predicative symbols, we can always rewrite $\Pi$ to a model equivalent program in functional normal form. So as in (Greco, Molinaro, and Trubitsyna 2013; Eiter and Simkus 2009), in the rest of this paper, we will only consider programs in their functional normal form. Also, for a given program $\Pi$, a finite set $D$ of facts ($D$ can be empty) is called an input database of $\Pi$ when we consider program $\Pi \cup D$.

GMT-bounded and Size-restricted Programs

Now we introduce the notions of gmt-bounded and size-restricted programs as proposed in (Greco, Molinaro, and Trubitsyna 2013) and (Calautti et al. 2015), respectively.

Generally speaking, for a given program $\Pi$, the gmt-boundedness and size-restrictedness are defined through two operators $\Psi_{\text{gm}(\Pi)} : 2^{\text{arg}(\Pi)} \rightarrow 2^{\text{arg}(\Pi)}$ and $\Psi_{\text{sr}(\Pi)} : 2^{\text{arg}(\Pi)} \rightarrow 2^{\text{arg}(\Pi)}$, respectively, such that for a given $A \subseteq \text{arg}(\Pi)$, we say that $\Pi$ is $A$-gmt-bounded (resp. $A$-size-restricted) iff $\Psi_{\text{gm}(\Pi)}(A) = \text{arg}(\Pi)$ (resp. $\Psi_{\text{sr}(\Pi)}(A) = \text{arg}(\Pi)$), for all $i \geq 0$, $\Psi_{\text{gm}(\Pi)}^i(A)(\Psi_{\text{sr}(\Pi)}^i(A)(A)$, resp.) is defined inductively as follows: (1) $\Psi_{\text{gm}(\Pi)}(A) = A$; (2) $\Psi_{\text{gm}(\Pi)}^i(A) = \Psi_{\text{gm}(\Pi)}(\Psi_{\text{gm}(\Pi)}^{i-1}(A)(A)$, resp.) denotes the $i$-greatest restricted arguments of $\Pi$, resp.; (2) $\Psi_{\text{sr}(\Pi)}^i(A) = \Psi_{\text{sr}(\Pi)}(\Psi_{\text{sr}(\Pi)}^{i-1}(A))$, $\Psi_{\text{sr}(\Pi)}^i(A)$, resp.) denotes the $i$-greatest restricted arguments of $\Pi$, resp.)

We say that a program $\Pi$ is gmt-bounded iff $\Psi_{\text{gm}(\Pi)}(\text{arg}(\Pi)) = \text{arg}(\Pi)$, where AR(II) denotes the set of restricted arguments (Lierler and Lifschitz 2009). On the other hand, although size-restrictedness specified in (Greco, Molinaro, and Trubitsyna 2013; Calautti et al. 2015) does not directly capture gmt-boundedness, it can nevertheless be incorporated through an “adornment” process achieved by making the set of input argument $A$ in $\Psi_{\text{sr}(\Pi)}(A)$ the gmt-bounded arguments. As such, in this paper, when we say that a program $\Pi$ is size-restricted, we mean that $\Psi_{\text{sr}(\Pi)}(A)$ is $\text{arg}(\Pi)$ such that $A$ is the gmt-bounded arguments of $\Pi$. It can be showed that this

\footnote{We refer readers to (Calautti et al. 2015) for more details about “A-size-restrictedness.”}
simple program \( \{ p(f(X,Y), Z) \leftarrow p(X, g(Z), g(Y)) \} \) is size-restricted (Calautti et al. 2015).

**An Upper Bound of Progression**

Our idea of discovering a new decidable class of programs is described as follows: for a given program \( \Pi \), (1) we firstly propose a progression based procedure to specify an approximating upper bound \( \Gamma(\Pi) \) for all stable models \( S \) of program \( \Pi \cup D \) for all input database \( D \); and (2) by imposing a proper polynomial bound on \( \Gamma(\Pi) \), we eventually are able to derive a new decidable class of programs with function symbols which are finitely ground when the corresponding bounds are met by these programs.

For a program \( \Pi \), \( \Pi^{\text{DEF}} \), we denote the program obtained from \( \Pi \) via the following transformation:

\[
\begin{align*}
(1) & \quad Hd(r') \leftarrow \text{Pos}(r') \wedge \neg\text{Neg}(r') \\
(2) & \quad \text{Pos}(r') = \text{Pos}(r) \\
(3) & \quad \text{Neg}(r') = \text{Neg}(r) \cup \{ \text{Hd}(r') \setminus \{ \text{Hd}(r') \} \}.
\end{align*}
\]

Intuitively, \( \Pi^{\text{DEF}} \) is the normal program obtained from \( \Pi \) by “shifting” (Dix, Gottlob, and Marek 1996). That is, for each rule of \( \Pi \), only one atom occurring in the head of this rule, while all the other head atoms are shifted to the negative body of the new generated rule.

Now based on programs \( \Pi \) and \( \Pi^{\text{DEF}} \), from the Herbrand base of \( \Pi \), we specify a procedure that computes the set of all facts that must be true or false derived from \( \Pi \).

**Definition 1. [Deriving lower bound] Let \( \Pi \) be a program. Then \( \mathcal{K}^k(\Pi) \) (\( k \geq 0 \)) is inductively defined as follows:

\[
\begin{align*}
\mathcal{K}^0(\Pi) & = \{ (s^+, -) \mid \text{there exists a rule } \alpha \leftarrow \top \in \Pi \} \\
\mathcal{K}^{k+1}(\Pi) & = \mathcal{K}^k(\Pi) \cup \\
& \{ (\alpha_1, \theta, -) \mid \text{there exist rules } \alpha_1 \leftarrow \beta_1, \beta_1^2 \in \Pi^{\text{DEF}} \text{ and mappings } \theta_1(i = 1, 2) : \theta_1 : \text{VAR}(\alpha_1) \rightarrow \text{VARCON}(\mathcal{K}^k(\Pi)) \} \cup \\
& \{ (\alpha_1, \theta, +) \mid \text{there exist rules } \alpha_1 \leftarrow \beta_1, \beta_1^2 \in \Pi^{\text{DEF}} \text{ and mappings } \theta_1(i = 1, 2) : \theta_1 : \text{VAR}(\alpha_1) \rightarrow \text{VARCON}(\mathcal{K}^k(\Pi)) \text{ such that:} \\
& (1) \alpha_1 \beta_1 = \alpha_2 \beta_2 \text{ and } (\beta_1 \beta_2 \notin \{ \beta_1, \beta_2 \} = \emptyset); \\
& (2) \beta_1^2 \beta_2 \beta_1^2 \in \mathcal{K}^k(\Pi) \text{ and } \beta_1^2 \beta_2 \beta_1^2 \in \mathcal{K}^k(\Pi) \}; \\
& \{ (\alpha_1, \theta, -) \mid \text{there exist rules } \alpha_1' \leftarrow \beta_1, \beta_1' \in \Pi^{\text{DEF}} \text{ and mappings } \theta_1(i = 1, 2) : \theta_1 : \text{VAR}(\alpha_1) \rightarrow \text{VARCON}(\mathcal{K}^k(\Pi)) \text{ such that:} \\
& (1) \alpha_1 \beta_1 = \alpha_2 \beta_2 \text{ and } (\beta_1 \beta_2 \notin \{ \beta_1, \beta_2 \} = \emptyset); \\
& (2) \beta_1^2 \beta_2 \beta_1^2 \in \mathcal{K}^k(\Pi) \text{ and } \beta_1^2 \beta_2 \beta_1^2 \in \mathcal{K}^k(\Pi); \\
& (3) \alpha_1' \beta_1 \in \mathcal{K}^k(\Pi^-) \text{ and } \alpha_2' \beta_2 \in \mathcal{K}^k(\Pi^-) \}, \quad (3)
\end{align*}
\]

Let \( \mathcal{K}^{\infty}(\Pi) = \bigcup_{i=0}^{\infty} \mathcal{K}^i(\Pi) \).

**Example 1.** Let \( \Pi \) be a program consisting of the following rules:

\[
\begin{align*}
r_0 &: \quad \bot \leftarrow q(X), \\
r_1 &: \quad \bot \leftarrow \neg p(X), \\
r_2 &: \quad p(f(X)) \leftarrow p(X), r(X), \neg q(X), \\
r_3 &: \quad r(X) \leftarrow r(f(X)), \\
r_4 &: \quad \bot \leftarrow r(X), p(g(X)), \neg q(X), \\
r_5 &: \quad \bot \leftarrow r(Y), \neg p(g(Y)), \neg q(Y).
\end{align*}
\]

Then according to Definition 1, we have:

\[
\begin{align*}
\mathcal{K}^0(\Pi) & = \{ (q(X), -), (q(Y), -), (r(X), -) \}; \\
\mathcal{K}^1(\Pi) & = \{ (r(f(X)), -), (r(f(Y)), -) \}; \\
\mathcal{K}^2(\Pi) & = \mathcal{K}^1(\Pi) \cup \{ \{ (q(X), -), (q(Y), -) \}; \\
\mathcal{K}^{\infty}(\Pi) & = \mathcal{K}^2(\Pi).
\end{align*}
\]

Now based on Definition 1, we define an upper bound for a given program \( \Pi \) as follows.

**Definition 2. [Deriving upper bound] Let \( \Pi \) be a program and \( S \subseteq \mathcal{K}^{\infty}(\Pi) \). Then \( \Gamma(\Pi)(S) \) (\( \geq 0 \)) is inductively defined as follows:

\[
\begin{align*}
\Gamma(\Pi)(S) & = \{ \text{Hd}(r) \mid \exists r \in \Pi^{\text{DEF}} \text{ and mapping } \theta : \text{VAR}(r) \rightarrow \text{VARCON}(\Pi) \text{ such that:} \\
& (1) \text{Pos}(r) \cap S^- = \emptyset; \\
& (2) \text{Neg}(r) \cap S^+ = \emptyset \}; \\
\Gamma^{i+1}(\Pi)(S) & = \Gamma(\Pi)(S) \cup \\
& \{ \text{Hd}(r) \mid \exists r \in \Pi^{\text{DEF}} \text{ and mapping } \theta : \text{VAR}(r) \rightarrow \text{VARCON}(\Pi) \text{ such that:} \\
& (1) \text{Pos}(r) \subseteq \Gamma(\Pi)(S); \\
& (2) \text{Neg}(r) \cap S^+ = \emptyset; \\
& (3) \text{Neg}(r) \cap S^- = \emptyset \}. \quad (6)
\end{align*}
\]
Finally, we define \( \Gamma_{II}^\infty(S) = \bigcup_{i=0}^\infty \Gamma_{II}^i(S) \) to be its fixpoint.

From (5) in Definition 2, we consider \( Hd(r) \) and mapping \( \theta \) only for those rules whose negative or positive bodies are not “defeated” yet by the sets \( S^+ \) and \( S^- \) (where \( S \subseteq K^\infty(II) \)), respectively. Then inductively, we have that (6) simply extends those derived “types” of atoms based on the ones obtained from previous steps.

**Theorem 1. [Upper bound for stable models]** Let \( \Pi \) be a program and \( S \subseteq K^\infty(\Pi) \). Then for every input database \( D \) and stable model \( A \) of \( \Pi \cup D \), \( A \preceq \Gamma_{II}^\infty(S) \).  

Proof. For convenience, given a set of atoms \( S' \subseteq \Gamma_{II}^\infty(S) \), denote \( S' \upharpoonright_{\text{CONST}(\Pi \cup D)} \) as the set of ground atoms \( \{ \alpha \theta \mid \alpha \in S' \text{ and } \theta \in \text{VAR}(\alpha) \rightarrow \text{CONST}(\Pi \cup D) \} \). Thus to prove this result, it will be sufficient to show that \( A \subseteq \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \) for any stable model \( A \) of \( \Pi \cup D \), because it then follows that \( A \preceq \Gamma_{II}^\infty(S) \).

On the contrary, assume that \( A \) is a stable model of \( \Pi' = \text{GROUND}(\Pi \cup D) \) such that \( A \nsubseteq \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \). Then since \( A \) is a stable model of \( \Pi' \), it follows that \( A \) is a minimal model of \( \Pi' A \) (where \( \Pi' A \) denotes the reduct of \( \Pi' \) on \( A \)). Let \( A' = A \setminus \{ a \in A \mid a \notin \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \} \), i.e., \( A' \) is the set obtained from \( A \) by deleting all the atoms that are not mentioned in \( \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \). Then since \( A \nsubseteq \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \) (which implies that \( \exists a \in A \) such that \( a \notin \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \)), it follows that \( A' \subset A \) where \( A' \subseteq \Gamma_{II}^\infty(S) \upharpoonright_{\text{CONST}(\Pi \cup D)} \). Then we can prove the result: \( A' \models (\Pi')^A \) (we omit the full proof of this result here due to a space limit), from which and the fact that \( A' \subset A \), we conclude a contradiction that \( A \) is a minimal model of \( (\Pi')^A \). \[ \square \]

**Polynomially Bounded Programs**

According to Theorem 1, it is clear that if \( \Gamma_{II}^\infty(S) \) (for some \( S \subseteq K^\infty(\Pi) \)) is a finite set, then \( \Pi \) is finitely ground. Also, from Definitions 1 and 2, we can see that if for each atom in \( \Gamma_{II}^\infty(S) \), its term depth is bounded by a fixed integer, then \( \Gamma_{II}^\infty(S) \) must be a finite set. For a given set \( A \) of atoms, let \( \text{DEP}(A) \) denote the set of all maximum term depths of all arguments \( p[i] \) mentioned in \( A \). Now our attempt is to impose a bound \( B \) on \( \text{DEP}(\Gamma_{II}^\infty(S)) \) such that \( \text{DEP}(\Gamma_{II}^\infty(S)) = \text{DEP}(\Gamma_{II}^\infty(S)) \). In this section, we will identify a new class of programs called polynomially bounded programs by defining a term depth bound for \( \Gamma_{II}^\infty(S) \) to be a polynomial in the size of program \( \Pi \).

Firstly, given a program \( \Pi \), we define

\[ I(\Pi) = N + N^3, \]

where \( N = |\Pi|^{\text{DEF}} \cdot \text{MAXART}(\Pi) \cdot \text{MAXPOS}(\Pi) \). Here \( \text{MAXPOS}(\Pi) \) denotes the maximum number of atoms in the positive body of a rule in \( \Pi \), and \( \text{MAXART}(\Pi) \) denotes the product of the maximum arities of a predicate and function symbols occurring in \( \Pi \), i.e., \( \text{MAXART}(\Pi) = m \times n \), where \( m \) and \( n \) are the maximum arities of the predicates and function symbols occurring in \( \Pi \), respectively.

Intuitively, \( I(\Pi) \), as defined via (7), gives an approximation of the minimum bound on the number of iterations of \( \Gamma_{II}^\infty(S) \) that has to be done in order to determine if an infinite propagation of terms may actually take place. In a nutshell, iterating through \( \Gamma_{II}^\infty(S) \), for \( 1 \leq k \leq I(\Pi) \), considers all the possible “transpositions” and “propagations” of an argument, say \( p[i] \), within the program \( \Pi \) as we iterate through each step.

At the same time, such iteration will compute the maximum possible depth of any restricted arguments, as well as the possible “undoing” of these complex terms. (i.e., the failing cycles). Indeed, from the definition of \( I(\Pi) \) in (7), the factor \( \text{MAXART}(\Pi) \) considers all the possible transpositions of \( p[i] \) within the arities of predicates and functions. In addition, the number \( |\Pi|^{\text{DEF}} \cdot \text{MAXPOS}(\Pi) \) also factors in the possible transposition that can be propagated through each positive atom in the program.

In fact, the number of iterative steps \( I(\Pi) \) does three things: (1) the number \( N = |\Pi|^{\text{DEF}} \cdot \text{MAXART}(\Pi) \cdot \text{MAXPOS}(\Pi) \) considers the iterative steps required to generate the deepest term of a restricted argument because it bounds the length of the greatest possible path that can derive a complex term of a restricted argument; (2) the additional number \( N^3 \) further adds the additional steps that are required to “undo” the complex terms compounded in the restricted arguments from doing the aforementioned first \( N \) steps because it is the product of the number of the deepest possible term (bounded by \( N \)) with that of the maximum cycle length (also bounded by \( N \)) that can “undo” the complexity of the term and where the one more factor of \( N \) (which makes the term “cubed” in (7)) considers the possibility that each can take \( N \)-steps to exhaust each of the possible positions of arguments in atoms; and (3) the fact that doing \( I(\Pi) \)-steps considers all the possible transpositions and propagations of an argument allows us to detect any growing cycles corresponding to unlimited growth of complex terms within the argument as well since it will also allow us to detect recursive information about function applications.

Given a set \( A \) of atoms, we denote by \( \text{DEP}(p[i], A) \) as the maximum term depth of the argument \( p[i] \) mentioned in \( A \), where we define \( \text{DEP}(p[i], A) = 0 \) if \( A = \emptyset \). Now, given a predicate \( p \in \text{PRED}(\Pi) \), denote by \( \text{POLYLA}_p(\Pi) \) as the set of arguments:

\[ \{ p[i] \mid \text{DEP}(p[i], (\Gamma_{II}^\infty(S))) = \text{DEP}(p[i], (\Gamma_{II}^\infty(S))) \}, \]

where \( S = \mathcal{K}^\infty(\Pi) \), i.e., the set of arguments of the predicate \( p \) that does not grow beyond the stage of iterations \( I(\Pi) \) of \( \Gamma_{II}^\infty(S) \) with \( S \) to be the set of non-ground atoms obtained from \( \mathcal{K}^\infty(\Pi) \). More generally, we call the arguments in the set \( \bigcup_{p \in \text{PRED}(\Pi)} \text{POLYLA}_p(\Pi) \) as the POLY-limited arguments of \( \text{ARG}(\Pi) \), which we denote by \( \text{POLYLA}(\Pi) \), i.e., just omitting the subscript of the particular predicate \( p \).

**Definition 3. [POLY-bounded programs]** Given a program \( \Pi \), we say that \( \Pi \) is polynomially bounded, or simply called POLY-bounded, iff \( \text{POLYLA}(\Pi) = \text{ARG}(\Pi) \).
Intuitively, Definition 3 says that if a program is POLY-bounded, then we have that all functional arguments cannot grow beyond the number of \( 2 \cdot T(\Pi) + 1 \) iterations of \( \Gamma_{k}^{\Pi}(s) \), where \( S = K^{\Pi}(\Pi) \). Note that Definition 3 defines the polynomial bound \( 2 \cdot T(\Pi) + 1 \) for computing \( \Gamma_{k}^{\Pi}(s) \), instead of \( T(\Pi) + 1 \). This is due to the possibility that the growth of term depth in \( \Pi \) may run through multiple arguments, from which we may only gain sufficient information to predict if the iteration will continue or stop, by computing the second run of iterations through all arguments.

**Example 2.** Consider program \( \Pi_1 \) we discussed in Section 1. We first rewrite this program to its functional normal form \( \Pi'_1 \) as follows:

\[
\begin{align*}
  r_1 & : \text{viewLarge}(X, Y) \lor \text{viewThumbnail}(\text{next}(X), Y) \rightarrow \text{viewThumbnail}(X, Y), \\
  r_3 & : \perp \rightarrow \text{viewThumbnail}(\text{next}(X), Y), \\
  r_4 & : \text{viewThumbnail}(X, Y) \rightarrow \text{viewThumbnail}(\text{next}(X), Y).
\end{align*}
\]

Then from \( r_3 \) and \( r_4 \) we can conclude that \( \text{viewThumbnail}(\text{next}(X), Y) \), \( \text{viewThumbnail}(\text{next}(X), Y) \in S = K^{\Pi}(\Pi'_1) \). Thus, it follows from Definitions 2 and 3 that the growth of the term in the argument \( \text{viewThumbnail}(\Pi'_1) \) as propagated via rule \( r_1 \) is bounded, which implies POLYA(\( \Pi'_1 \)) = ARG(\( \Pi'_1 \)). That is, \( \Pi'_1 \) is POLY-bounded. \( \square \)

**Proposition 1.** If \( \Pi \) is POLY-bounded, then for every input database \( D \) (\( D \) can be empty), program \( \Pi \in D \) is finitely ground.

**Theorem 2.** GMT-bounded \( \subseteq \) POLY-bounded, and SR \( \subseteq \) POLY-bounded, where GMT-bounded, SR and POLY-bounded denote the three classes of GMT-bounded, size-restricted and POLY-bounded programs, respectively.

**Computational Complexity**

In this section, we study the complexity properties of POLY-bounded programs. We are mainly interested in the complexity of membership decision problem for this class of programs. Firstly, we have proved the following result for GMT-bounded and size-restricted programs.

**Proposition 2.** Deciding whether a program \( \Pi \) is GMT-bounded or size-restricted is in PSPACE.

**Theorem 3.** Deciding whether a program \( \Pi \) is POLY-bounded is EXPTIME-complete. The hardness holds even if \( \Pi \)'s maximum function arity is 2.

**Proof.** (Sketch) The membership can be obtained by outlining a membership decision procedure based on Definitions 1, 2 and 3.

Here we provide the main idea and procedure of proving the hardness. Let \( L \) be an arbitrary decision problem in EXPTIME. Then from the definition of complexity class EXPTIME (Papadimitriou 1994), there exists some deterministic Turing machine \( M \) such that for any string \( s, s \in L \) iff \( M \) accepts \( s \) in at most \( 2^{p(|s|)} \) steps for some polynomial \( p(n) \). Thus, assume the Turing machine \( M \) to be the tuple \( (Q, \Gamma, \sqcup, \Sigma, \delta, q_0, F) \) such that: (1) \( Q \neq \emptyset \) is a finite set of states; \( (2) \Gamma \neq \emptyset \) is a finite set of alphabet symbols; \( (3) \sqcup \in \Gamma \) is the “blank” symbol; \( (4) \Sigma \subseteq \Gamma \setminus \{ \sqcup \} \) is the set of input symbols; \( (5) \delta: (Q \setminus F) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\} \) is the transition function; \( (6) q_0 \in Q \) is the initial state; and lastly, \( (7) F \subseteq Q \) is the set of final/accepting states.

Now given a string \( s = a_0 \ldots a_{|s|-1} \) such that \( a_i \in \Sigma \) for \( 0 \leq i < |s| \), we construct a program4 \( \Pi_{M(s)} = \Pi_{\text{ORD}}^{\Pi_{M(s)}} \cup \Pi_{\text{STR}}^{\Pi_{M(s)}} \cup \Pi_{\text{ASSOC}}^{\Pi_{M(s)}} \cup \Pi_{\text{EDGES}}^{\Pi_{M(s)}} \cup \Pi_{\text{TRANS}}^{\Pi_{M(s)}} \cup \Pi_{\text{ACCEPT}}^{\Pi_{M(s)}} \cup \Pi_{\text{BOUND}}^{\Pi_{M(s)}} \).

Program \( \Pi_{\text{ORD}}^{\Pi_{M(s)}} \) is to generate the linear ordering on the \( p(|s|) \)-length tuples that will encode the computation time/steps as well as the individual cell-positions in the tape.

Program \( \Pi_{\text{STR}}^{\Pi_{M(s)}} \) generates all possible strings of lengths from 1 to \( \Gamma^{|p(|s|)|} \) under the alphabets of \( \Gamma \), as defined in the following:

\[
\begin{align*}
\Pi_{\text{STR}}^{\Pi_{M(s)}} &= \\
\{ s_0^0(a_0, t, t) & \leftarrow T \ | \ 0 \leq i \leq n - 1 \} \cup \{ s_0^0(\sqcup, t, T) & \leftarrow T - t \} \cup \{ s_0^0(X \circ Y, T_1, T_4) & \leftarrow s_0^0(X, T_1, T_2), s_0^0(Y, T_3, T_4), \} \cup \{ T_1 \leq T_2, T_3 < T_4, T_0 \} \cup \{ 0 \leq i, j < k \leq p(|s|) \} \cup \{ s^0(a, T_1, T_3) & \leftarrow \text{num}(T_1), \ldots, \text{num}(T_{p(|s|)}) \} \cup \{ T_1 \leq T_2, T_2 < T_3, T_3 \leq T_4 \} \cup \{ 0 \leq i, j, k \leq p(|s|) \} \cup \{ s_0^0(X, T_1, T_2) & \leftarrow s_0^0(X, T_1, T_2), \} \cup \{ s(X, T_1, T_2) & \leftarrow s^0(X, T_1, T_2) \ | \ 0 \leq i \leq p(|s|) \}. \}
\end{align*}
\]

Without loss of generality, we assume \( |\Gamma| > 2 \), therefore, it is sufficient to use strings of length from 1 to \( |\Gamma^{|p(|s|)|} | \) to encode all possible \( 2^{|\Pi_{M(s)}|} \) configurations. In program \( \Pi_{\text{STR}}^{\Pi_{M(s)}} \), we define the function “o” taking arguments of two strings \( s_1 \) and \( s_2 \) so that \( s_1 \circ s_2 \) denotes the concatenation of \( s_1 \) and \( s_2 \). Then due to the necessity rules (10) and (12), it is observed that it would only take \( O(p(|s|)) \)-steps to generate all such strings of lengths from 1 to \( |\Gamma^{|p(|s|)|} | \).

Here predicate \( s_0^0 \) is used to represent the input string on the tape, and predicates \( s^0_i \ (0 \leq i \leq p(|s|)) \) are used to generate such initial string; while predicate \( s \) represents an arbitrary string on the tape, which is generated from predicates \( s^0_i \ (0 \leq i \leq p(|s|)) \).

The program \( \Pi_{\text{STR}}^{\Pi_{M(s)}} \), on the other hand, encodes the string associative property axioms that are needed by the concatenation function “o” (here we omit the definition of this program).

4Due to a space limit, below we only provide the major program definitions.
Programs $\Pi^{\text{EDGES}}_{M(s)}$ and $\Pi^{\text{TRANS}}_{M(s)}$ described below then encode the machine $M(s)$’s configuration changes based on the corresponding state transitions in $M(s)$, for that we view that the input string $s$ is accepted by machine $M(s)$ as the problem of reachability from the initial configuration of $M(s)$ to $M(s)$’s final (accepting) configuration.

$$\Pi^{\text{EDGES}}_{M(s)} = \begin{cases} cf \left( X_t, q, X \circ o a, \bar{O}, X_{tp} \mid c \circ Y, Y_{tp}, \bar{N} \right) \\ \vdash cf \left( Y_t, q’, X \circ o b c, \bar{O}, Y_{tp} \mid Y, Z_{tp}, \bar{N} \right) \\ X_t \prec Y_t, X_{tp} \prec Y_{tp}, c \circ o Y_{tp} \prec Z_{tp}, \\ s(X \circ o a, \bar{O}, X_{tp}), s(c \circ Y, Y_{tp}, \bar{N}), \\ s(X \circ o b c, \bar{O}, Y_{tp}), s(Y, Z_{tp}, \bar{N}), \end{cases}$$

$$| \delta(q, a) = (q’, b, R), c \in \Gamma \text{ and } N = |\Gamma|^{|p(s)|} \} \cup (14)$$

$$\Pi^{\text{TRANS}}_{M(s)} = \begin{cases} cf(X_t, q, X \circ o a, \bar{O}, X_{tp} \mid Y, Y_{tp}, \bar{N}) \\ \vdash cf(Y_t, q’, X \circ o b c, \bar{O}, Y_{tp} \mid Y, Z_{tp}, \bar{N}) \\ X_t \prec Y_t, Z_{tp} \prec Y_{tp}, c \circ o Y_{tp} \prec Z_{tp}, \\ s(X \circ o a, \bar{O}, X_{tp}), s(Y, Y_{tp}, \bar{N}), \\ s(X \circ o b c, \bar{O}, Z_{tp}), s(b \circ Y, X_{tp}, \bar{N}) \end{cases}$$

$$| \delta(q, a) = (q’, b, L), c \in \Gamma \text{ and } N = |\Gamma|^{|p(s)|} \}; (15)$$

Here notation “$cf \left( X_t, q, X \circ o a, \bar{O}, X_{tp} \mid Y, Y_{tp}, \bar{N} \right)$” mentioned in (14), and denoted as “$cf \left( X \right)$” in (16), represents the machine’s configuration. For a space reason, we omit the detailed explanation on this encoding. The expression

$$“cf \left( X_t, q, V, \bar{O}, X_{tp} \mid W, Y_{tp}, \bar{N} \right)” \vdash cf \left( Y_t, q’, X \circ o b c, \bar{O}, Y_{tp} \mid Y, Z_{tp}, \bar{N} \right),$$

(17)

denotes a relation encoding a configuration change, for a given state transition $\delta(q, a) = (q’, b, R)$ in $M(s)$. The expression “$cf \left( Y \right) \vdash cf \left( Y’ \right)$” in (16) encodes the transitive extension of “$=$”, for which it is read: configuration “$cf \left( Y’ \right)$” is reached from configuration “$cf \left( Y \right)$”.

What program $\Pi^{\text{EDGES}}_{M(s)}$ does is to establish the initial connections between any two configurations based on the input state transitions from $M(s)$, which we call edges. For instance, suppose $M(s)$ accepts string $s$ in $2^{|p(s)|}$ steps, through the sequence of configuration changes: $cf_0, cf_1, \ldots, cf_{2^{|p(s)|} - 1}$, then $\Pi^{\text{EDGES}}_{M(s)}$ will establish edges $cf_0 \vdash cf_1, cf_1 \vdash cf_2, \ldots, cf_{2^{|p(s)|} - 1} \vdash cf_{2^{|p(s)|} - 1}$. Then the transitive closure of $\vdash$, which is defined based on $\vdash$ through transitive rules in $\Pi^{\text{TRANS}}_{M(s)}$, is derived by the following manner: firstly, the reachability between any two configurations within two steps is derived: $cf_0 \vdash cf_2, cf_1 \vdash cf_3, cf_2 \vdash cf_4, \ldots, cf_{2^{|p(s)|} - 3} \vdash cf_{2^{|p(s)|} - 1}$, then in the second run of the evaluation, the reachability between any two configurations within four steps are derived: $cf_0 \vdash cf_4, cf_1 \vdash cf_5, cf_3 \vdash cf_6, \ldots, cf_{2^{|p(s)|} - 5} \vdash cf_{2^{|p(s)|} - 1}$. This process continues until the reachability from $cf_0$ to $cf_{2^{|p(s)|} - 1}$, i.e., $cf_0 \vdash cf_{2^{|p(s)|} - 1}$, is derived. As we will prove in Lemma 1, $cf_0 \vdash cf_{2^{|p(s)|} - 1}$ will be derived within polynomial steps iff $M(s)$ accepts $s$ in $2^{|p(s)|}$ steps.

Finally, we have

$$\Pi^{\text{ACCEPT}}_{M(s)} = \begin{cases} \text{accept } \iff cf \left( \bar{O}, q_0, a_0, \bar{O}, a_1 \circ O, X, Y_{tp}, \bar{N} \right) \vdash cf \left( X_t, q, Y, \bar{O}, X_{tp} \mid Z, Y_{tp}, \bar{N} \right), \\ \text{true} \end{cases}$$

$$| q \in F \text{ and } N = |\Gamma|^{|p(s)|} \}; (18)$$

$$\Pi^{\text{BOUND}}_{M(s)} = \{ r(f (X)) \rightarrow r(X), \text{not accept} \}, (19)$$

where $\Pi^{\text{ACCEPT}}_{M(s)}$ simply derives the propositional atom “accept” if a configuration at an accepting state in $F \subseteq Q$ can be reached from the initial configuration under the input string represented via predicate $s_0$; otherwise, program $\Pi^{\text{BOUND}}_{M(s)}$ will make the entire program $\Pi_{M(s)}$ unbounded. Then we can prove the following lemma:

**Lemma 1.** $M$ accepts $s$ iff $\Gamma^3(\Pi_{M(s)})(S) = \Gamma^2(\Pi_{M(s)})(S)$, i.e., $\Pi_{M(s)}$ is POLY-bounded, where $\Gamma(\Pi_{M(s)}) = O(|p(s)|^{10})$, and $S = K^2(\Pi_{M(s)})$.

$$\square$$

According to Theorem 3, we can see that compared to other decidable classes, the membership of POLY-bounded programs requires extra computations, which is consistent with the fact that this class of programs strictly contains all previous decidable classes.

**Concluding Remarks**

The problem of logic program termination has been extensively studied under the top-down evaluation approach over the years: (Schreie and Decorte 1994; Voets and Schreye 2011; Marchiori 1996; Ohlebusch 2001; Genaim and Codish 2005; Serebrenik and Schreye 2005; Nishida and Vidal 2010; Schneider-Kamp, Giesl, and Nguyen 2009; Schneider-Kamp et al. 2009; 2010; Nguyen et al. 2007; Brunskekle and Schreye 2005; Basileci and Bonatti 2004; Basileci and Criscuolo 2009b; Zhang, Zhang, and You 2015). However, as pointed by Greco, et al. (2013), under the stable model semantics, these methods are not generally applicable to identify finitely ground programs.

In this paper, by proposing the stable model polynomial upper bound for logic programs, we discovered a new decidable class of finitely ground programs, which strictly contains the other newly identified decidable classes named GMT-bounded and size-restricted programs (Greco, Molinaro, and Trubutsyna 2013; Calautti et al. 2015).


References


