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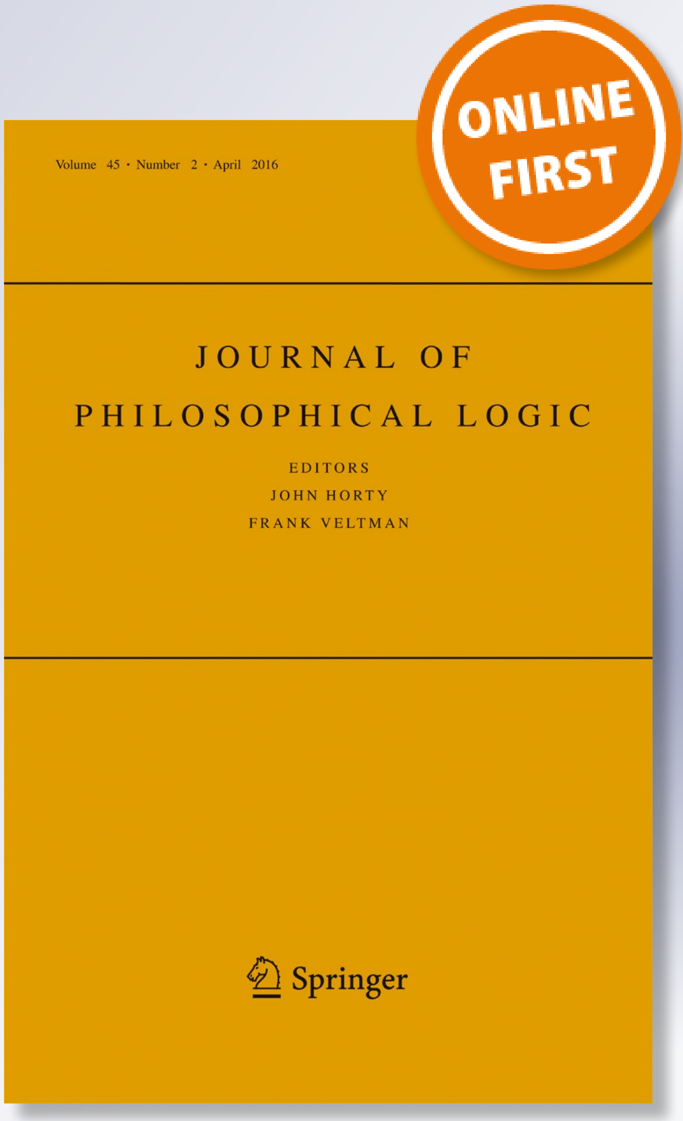

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Inter-Definability of Horn Contraction and Horn Revision

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Abstract There have been a number of publications in recent years on generalising the AGM paradigm to the Horn fragment of propositional logic. Most of them focused on adapting AGM contraction and revision to the Horn setting. It remains an open question whether the adapted Horn contraction and Horn revision are inter-definable as in the AGM case through the Levi and Harper identities. In this paper, we give a positive answer by providing methods for generating contraction and revision from their dual operations. Noticeably, we cannot apply the Levi and Harper identities directly in such methods as the Horn fragment does not fully support negation. To overcome this difficulty, a Horn approximation technique called Horn strengthening is used. We show that Horn contraction generated from Horn revision is always plausible whereas Horn revision generated from Horn contraction is, in general, implausible and, to regain plausibility, the generating contraction has to be properly restricted.

Keywords Belief change · Horn logic · Non-monotonic reasoning

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1 Introduction

The theory of *belief change* deals with how an agent changes its beliefs in a rational manner when confronted with new information. The dominant approach is the so-called *AGM paradigm* [2, 15] which focuses on two kinds of changes, namely *contraction* and *revision*. The paradigm requires the full expressive power of propositional logic. This can be a limitation in artificial intelligence. Many useful applications are based on logics in which such expressive power is not available. Obviously, the AGM paradigm can not be applied to these applications directly. To overcome the limitation, significant effort has been made on *Horn belief change*, that is, to generalise the AGM paradigm to the *Horn fragment of propositional logic* (i.e., *Horn logic*) [1, 3–5, 8–12, 23, 32–36].

The AGM paradigm provides several construction methods for contraction and revision including *partial meet contraction* [2], *entrenchment-based contraction* [15, 16], and *model-based revision* [17, 21]. Other than constructing contraction and revision directly through one of the construction methods, it has long been discovered that contraction and revision can be constructed indirectly from one another via the so called *Harper identity* [20] and *Levi identity* [24].

According to Levi [24], to revise an agent's belief set K by a formula ϕ, we can first contract by the negation of ϕ then expand by ϕ.[1] Since the contraction removes formulas that potentially contradict with ϕ, the belief set obtained after the expansion is guaranteed to be consistent and contains ϕ. Formally, if $-$ is a contraction function for K and $+$ the expansion function,[2] then a revision function $*$ for K can be defined as $K * \phi = (K - \neg\phi) + \phi$ for all formulas ϕ. The revision function thus defined is an AGM revision function if and only if the generating contraction is an AGM contraction function.[3]

According to Harper [20], to contract an agent's belief set K by a formula ϕ, we can first revise by the negation of ϕ, then intersect with K. The revision produces a consistent belief set that implies the negation of ϕ, thus, after the intersection we get a subset of K that fails to imply ϕ. Formally, if $*$ is a revision function for K, then a contraction function $-$ for K can be defined as $K - \phi = (K * \neg\phi) \cap K$ for all formulas ϕ. The contraction function thus defined is an AGM contraction function if and only if the generating revision function is an AGM revision function.

While most works on Horn belief change concentrate on constructing Horn contraction and Horn revision by adapting the aforementioned construction methods to Horn logic, none has established a connection between Horn contraction and Horn revision as in AGM contraction and revision. In this paper, we aim to fill this gap by providing methods for generating plausible Horn contraction and revision functions from their dual operations.

[1] Levi's actual proposal is more general but this form has been adopted for the AGM paradigm.

[2] The expansion function $+$ takes a belief set K and a formula ϕ and returns the logical closure of their union.

[3] We refer to AGM revision (contraction) functions as those that satisfy the full set of AGM revision (contraction) postulates.

Currently, the only work on Horn revision is due to Delgrande and Peppas [9, 10], in which they take a model-theoretic approach. Therefore, it makes sense to focus on the model-theoretic approach in establishing connections between Horn contraction and Horn revision. For this reason, we will first provide a model-theoretic approach to defining Horn contraction. The model-based Horn contraction thus defined is general enough to subsume all existing Horn contractions assuming certain forms of plausibility ranking [34, 36], which gives rise to another reason for favouring the model-theoretic approach.

With both model-based Horn contraction and revision at our disposal, we will next investigate methods for generating them from one another. The main difficulty in generating such revision and contraction functions lies in Horn logic's inability to express negations for certain Horn formulas. In the intermediate contraction step of generating a revision function via the Levi identity, we need to contract by the negation of the revising formula. However, the negation of a Horn formula may be non-Horn and Horn contraction functions do not operate on non-Horn formulas. The problem also appears in generating contraction functions. To overcome this difficulty, we make use of the Horn approximations of the non-Horn negations. The intermediate contraction and revision step are then achieved by contracting and revising such Horn approximations.

Since often there are multiple Horn approximations for a non-Horn formula, we may need to contract and revise by multiple formulas in the intermediate contraction and revision step. One would immediately think of using package contraction and revision [14], however, as we are investigating inter-definability of singleton contraction and revision, package contraction and revision are not appropriate in this context. Therefore we focus on singleton contraction and revision and come up with two equivalent ways of contracting by multiple formulas through singleton contraction and one way of revising by multiple formulas through singleton revision which lead to two equivalent methods for generating model-based Horn revision functions from model-based Horn contraction functions and one method for generating model-based Horn contraction functions from model-based Horn revision functions.

The remainder of this paper is organised as follows. We first give the necessary technical preliminaries in Section 2, then we provide a brief overview in Section 3 on the model-theoretic approach to defining AGM contraction and revision. This is followed by details of the model-based Horn revision in Section 4. In Section 5, we introduce our model-based Horn contraction and present its connections with other Horn contractions. As the main results of this paper, we describe the generation of model-based Horn revision and contraction functions from their dual operations in Section 6 and 7. Finally, we conclude the paper in Section 8. All proofs are given in the appendix. Some results of this paper appear in our earlier work [35, 37].

2 Technical Preliminaries

We assume a propositional language $\mathcal{L}$ over a finite set of atoms $\mathcal{P}$ which is closed under the usual truth-functional connectives and contains the propositional constants $\top$ (truth) and $\bot$ (falsum). Atoms are denoted by lower case Roman letters $(p, q, \ldots)$.

Formulas are denoted by lower case Greek letters ($\phi, \psi, \ldots$). Sets of formulas are denoted by upper case Roman letters ($V, X, \ldots$). A *clause* is a disjunction of positive and negative atoms. A *Horn clause* is a clause with at most one positive atom. A *Horn formula* is a conjunction of Horn clauses.

The logic generated from $\mathcal{L}$ is specified by the standard Tarskian consequence operator Cn. For any set of formulas X, $Cn(X)$ denotes the set of formulas entailed by X. For any formula ϕ, $Cn(\phi)$ abbreviates $Cn(\{\phi\})$. We sometimes write $X \vdash \phi$ to denote $\phi \in Cn(X)$, $\phi \equiv \psi$ to denote $Cn(\phi) = Cn(\psi)$, and $\vdash \phi$ to denote $\phi \in Cn(\emptyset)$. The letter K is reserved to represent a *theory* or a *belief set* which is a set of formulas closed under logical deduction (i.e., $K = Cn(K)$).

Standard two-valued model-theoretic semantics is assumed. The set of all interpretations is denoted by Ω. An interpretation $\mu \in \Omega$ is a model of a formula ϕ if ϕ is true in μ, written $\mu \models \phi$. For any set of formulas X, $|X|$ denotes the set of models of X. For any formula ϕ, $|\phi|$ abbreviates $|\{\phi\}|$. For $\mathcal{P} = \{a, b, c, \ldots\}$, we write an interpretation as a bit vector $011\ldots$ to indicate that a is assigned false, b is assigned true, c is assigned true, etc. The theory operator $\mathcal{T} : 2^{\Omega} \to 2^{\mathcal{L}}$ is such that, given a set of interpretations M, $\mathcal{T}(M)$ is the set of formulas that are true in all interpretations of M. Formally, $\mathcal{T}(M) = \{\phi \in \mathcal{L} \mid \mu \models \phi \text{for every } \mu \in M\}$.

The Horn language $\mathcal{L}_{\mathrm{H}}$ is the subset of $\mathcal{L}$ containing only Horn formulas. The Horn logic generated from $\mathcal{L}_{\mathrm{H}}$ is specified by the consequence operator Cn_{H} such that, for any set of Horn formulas X, $Cn_{\mathrm{H}}(X) = Cn(X) \cap \mathcal{L}_{\mathrm{H}}$. The letter H is reserved to represent a *Horn theory* or a *Horn belief set* which is a set of Horn formulas such that $H = Cn_{\mathrm{H}}(H)$. The Horn subset function $\mathcal{H} : 2^{\mathcal{L}} \to 2^{\mathcal{L}_{\mathrm{H}}}$ is such that, given a set of formulas X, $\mathcal{H}(X)$ is the set of Horn formulas in X. Formally, $\mathcal{H}(X) = \{\phi \mid \phi \in X \text{ and } \phi \in \mathcal{L}_{\mathrm{H}}\}$. The Horn theory operator $\mathcal{T}_{\mathrm{H}} : 2^{\Omega} \to 2^{\mathcal{L}_{\mathrm{H}}}$ is such that, given a set of interpretations M, $\mathcal{T}_{\mathrm{H}}(M)$ is the set of Horn formulas that are true in all interpretations of M. Formally, $\mathcal{T}_{\mathrm{H}}(M) = \{\phi \in \mathcal{L}_{\mathrm{H}} \mid \mu \models \phi \text{ for every } \mu \in M\}$.

The *intersection* of two interpretations μ and ν is the interpretation, denoted as $\mu \cap \nu$, that assigns true only to those atoms that are assigned true by both μ and ν. For instance, if $\mu = 001$ and $\nu = 101$, then $\mu \cap \nu = 001$. Sometimes, we say μ and ν *induce* ω to indicate that ω is the intersection of μ and ν. Given a set of interpretations M, the closure of M under intersection is denoted as $Cl_{\cap}(M)$. Formally, $Cl_{\cap}(M) = \{\omega \mid \omega \in M \text{ or there are } \mu, \nu \in M \text{ such that } \mu \cap \nu = \omega\}$. The fundamental property of Horn formulas is that the set of models of any Horn formula is closed under $Cl_{\cap}$ and we call it *Horn closed*. Conversely, any Horn closed set of models corresponds to a unique Horn formula (modulo logical equivalence).

We present the following properties of the consequence operator Cn_{H} and the Horn closure operator $Cl_{\cap}$. These properties are essential in proving results in upcoming sections and we shall use them without referring to them explicitly.

Lemma 1 *Let H_1 and H_2 be sets of Horn formulas such that $H_1 = Cn_H(H_1)$ and $H_2 = Cn_H(H_2)$ and M, N sets of interpretations. Then*

1. $|H_1| \subseteq |H_2|$ *iff* $H_2 \subseteq H_1$
2. $H_1 \cap H_2 = Cn_H(H_1 \cap H_2)$
3. $|H_1 \cap H_2| = Cl_{\cap}(|H_1| \cup |H_2|)$

4. *If* $M \subseteq N$, *then* $Cl_\cap(M) \subseteq Cl_\cap(N)$
5. $Cl_\cap(Cl_\cap(M) \cup N) = Cl_\cap(M \cup N)$

In obtaining Horn approximations for non-Horn formulas, the notion of *Horn strengthening* will prove to be very useful. The notion was first proposed by Kautz and Selman [22] in the context of knowledge compilation. Their original definition is for clauses and sets of clauses such that a Horn strengthening for a clause ϕ is logically the weakest Horn clause that entails ϕ and a Horn strengthening for a set of clauses $\{\phi_1, \ldots, \phi_n\}$ is any set of Horn clauses $\{\phi'_1, \ldots, \phi'_n\}$ where ϕ'_i is the Horn strengthening of ϕ_i. Here we reformulate their definition to cover arbitrary formulas such that a Horn strengthening for a formula ϕ is logically the weakest Horn formula that entails ϕ.

Definition 1 The set of Horn strengthenings of ϕ, denoted as $\mathcal{HS}(\phi)$, is such that $\chi \in \mathcal{HS}(\phi)$ iff

1. $\chi \in \mathcal{L}_{\mathrm{H}}$
2. $|\chi| \subseteq |\phi|$
3. there is no $\chi' \in \mathcal{L}_{\mathrm{H}}$ such that $|\chi| \subset |\chi'| \subseteq |\phi|$

According to Definition 1, a Horn formula has exactly one Horn strengthening, namely itself. In the limiting case of ϕ being a tautology, we assume that its single Horn strengthening is itself. Since Definition 1 is model-theoretic, the sets of Horn strengthenings for logically equivalent formulas are identical. Also it is easy to show that the set of models of ϕ is equal to the union of the models of its Horn strengthenings.

Lemma 2 *If* $\mathcal{HS}(\phi) = \{\chi_1, \ldots, \chi_n\}$, *then* $|\phi| = |\chi_1| \cup \cdots \cup |\chi_n|$.

3 AGM Model-Based Contraction and Revision

The model-theoretic account of AGM revision is due to Grove [17] who constructs revision functions through *systems of spheres* motivated by a construction due to Lewis [25] for counterfactuals. Katsuno and Mendelzon [21] gave an equivalent formulation through pre-orders of interpretations which is obtained by restricting to a finite underlying propositional logic and considering a semantic counterpart to the systems of spheres.

In the account of Katsuno and Mendelzon [21], a *pre-order* $\preceq$ is a reflexive and transitive binary relation over Ω. The strict relation $\prec$ is defined in the standard way as $\mu \prec \nu$ if and only if $\mu \preceq \nu$ and $\nu \not\preceq \mu$. The equivalence relation $\simeq$ is also defined in the standard way as $\mu \simeq \nu$ if and only if $\mu \preceq \nu$ and $\nu \preceq \mu$. A pre-order $\preceq$ is *total* if, for every pair of $\mu, \nu \in \Omega$, either $\mu \preceq \nu$ or $\nu \preceq \mu$. For a set of interpretations M, $min(M, \preceq)$ represents the minimal elements of M by means of $\preceq$:

$$min(M, \preceq) = \{\mu \in M \mid \text{there is no } \nu \in M \text{ such that } \nu \prec \mu\}.$$

Each belief set K is assigned a pre-order $\preceq_K$ which represents a measure of closeness between the models of K and an interpretation such that $\mu \preceq_K \nu$ means μ is at least as close (i.e., plausible) to the models of K as ν. $\preceq_K$ is *faithful*[4] if

$$min(\Omega, \preceq_K) = |K|.$$

This definition captures the intuition that models of K are closest to themselves. The ordering over models also reflects an ordering of plausibility. The minimal models are considered the most plausible. Of the remaining models, those closer to the models of K are considered more plausible than those further away. In the following, we will omit the belief set subscript of $\preceq$ whenever it is clear from the context which is the associated belief set. In the revision of K by ϕ, a *model-based revision* function first obtains an intermediate model set that consists of models of ϕ that are the closest to those of K by means of the total faithful pre-order assigned to K. The theory operator is then applied to the models to obtain the revision outcome.

Definition 2 A function $*$ is a model-based revision function for K iff

$$K * \phi = \mathcal{T}(min(|\phi|, \preceq))$$

for all $\phi \in \mathcal{L}$, where $\preceq$ is a faithful total pre-order for K.

In the above definition, the pre-order $\preceq$ is referred to as the *determining* pre-order for $*$ and we also say $*$ is *determined* by $\preceq$. Also, we call K the *original belief set*, ϕ the *revising formula*, and $K * \phi$ the *revised belief set*. Model-based revision functions can be characterised by the full set of AGM revision postulates:

$(K * 1)$ $K * \phi = Cn(K * \phi)$
$(K * 2)$ $K * \phi \subseteq K + \phi$
$(K * 3)$ If $\perp \notin K + \phi$, then $K + \phi \subseteq K * \phi$
$(K * 4)$ $\phi \in K * \phi$
$(K * 5)$ If ϕ is consistent, then $\perp \notin K * \phi$
$(K * 6)$ If $\phi \equiv \psi$, then $K * \phi = K * \psi$
$(K * 7)$ $K * \phi \wedge \psi \subseteq (K * \phi) + \psi$
$(K * 8)$ If $\perp \notin (K * \phi) + \psi$, then $(K * \phi) + \psi \subseteq K * \phi \wedge \psi$

In the AGM tradition, $(K * 1)$–$(K * 6)$ are referred to as the *basic postulates* for revision and $(K * 7)$ and $(K * 8)$ are referred to as the *supplementary postulates* for revision.

It is well known in the belief change community that the model-theoretic account can also be applied in constructing contraction functions.

[4]In Katsuno and Mendelzon [21], a belief set is represented as a formula and faithfulness is a condition for the assignment of pre-orders to formulas. An assignment is *faithful* if logically equivalent formulas are assigned the identical pre-order and for any formula ϕ, its assigned pre-order $\preceq$ is such that $min(\Omega, \preceq) = |\phi|$. Here we work with a belief set directly, which is the same as working with formulas since the language $\mathcal{L}$ is finite. Also, we use faithfulness as a condition on the pre-orders to simplify the presentation.

Definition 3 A function $\dot{-}$ is a model-based contraction function for K iff

$$K \dot{-} \phi = \mathcal{T}(|K| \cup min(|\neg\phi|, \preceq))$$

for all $\phi \in \mathcal{L}$, where $\preceq$ is a faithful total pre-order for K.

Similar to revision, $\preceq$ is referred to as the determining pre-order for $\dot{-}$, K the *original belief set*, ϕ the *contracting formula*, and $K \dot{-} \phi$ the *contracted belief set*. In contracting ϕ, the contracted belief set $K \dot{-} \phi$ is obtained by applying the theory operator to the union of the minimal counter-models of ϕ and the models of K. Model-based contraction functions can be characterised by the full set of AGM contraction postulates:

$(K \dot{-} 1)$ $K \dot{-} \phi = Cn(K \dot{-} \phi)$
$(K \dot{-} 2)$ $K \dot{-} \phi \subseteq K$
$(K \dot{-} 3)$ If $\phi \notin K$, then $K \dot{-} \phi = K$
$(K \dot{-} 4)$ If $\not\vdash \phi$, then $\phi \notin K \dot{-} \phi$
$(K \dot{-} 5)$ $K \subseteq (K \dot{-} \phi) + \phi$
$(K \dot{-} 6)$ If $\phi \equiv \psi$, then $K \dot{-} \phi = K \dot{-} \psi$
$(K \dot{-} 7)$ $K \dot{-} \phi \cap K \dot{-} \psi \subseteq K \dot{-} \phi \wedge \psi$
$(K \dot{-} 8)$ If $\phi \notin K \dot{-} \phi \wedge \psi$, then $K \dot{-} \phi \wedge \psi \subseteq K \dot{-} \phi$

$(K \dot{-} 1)$–$(K \dot{-} 6)$ are referred to as the *basic postulates* for contraction and $(K \dot{-} 7)$ and $(K \dot{-} 8)$ are referred to as the *supplementary postulates* for contraction.

Since model-based contraction and revision functions are AGM contraction and revision functions, they are inter-definable. If $\dot{-}$ is a model-based contraction function for K, then the revision function $*$ defined as $K * \phi = (K \dot{-} \neg\phi) + \phi$ is a model-based revision function for K. If $*$ is a model-based revision function for K, then the contraction function $\dot{-}$ defined as $K \dot{-} \phi = (K * \neg\phi) \cap K$ is a model-based contraction function for K. Moreover, each model-based contraction function is definable from a model-based revision function and vice versa.

4 Defining Horn Revision Model-Theoretically

Delgrande and Peppas [9, 10] investigated the model-theoretic approach in defining Horn revision. They first proposed a construction which directly adapts the classic approach. This construction starts by obtaining an intermediate model set which consists of the minimal models of the revising formula, then returning the corresponding Horn belief set as the revision outcome. As in the classic approach, the minimal models are determined through a total and faithful pre-order over Ω. We refer to the construction as *model-based Horn revision* (MHR).

Definition 4 [9] A function $*$ is a MHR function for H iff

$$H * \phi = \mathcal{T}_{\mathrm{H}}(min(|\phi|, \preceq))$$

for all $\phi \in \mathcal{L}_{\mathrm{H}}$, where $\preceq$ is a faithful total pre-order for H.

We refer to models of the revised Horn belief set $H * \phi$ as the *resulting models* of the revision. Since $\mathcal{T}_{\mathrm{H}}(min(|\phi|, \preceq))$ is a Horn belief set, its models are Horn closed. That is

$$|H * \phi| = |\mathcal{T}_{\mathrm{H}}(min(|\phi|, \preceq))| = Cl_{\cap}(min(|\phi|, \preceq)).$$

Clearly, there is no guarantee that the minimal models of the revising formula (i.e., $min(|\phi|, \preceq)$) are Horn closed, thus the set of resulting models may contain non-minimal models of the revising formula.

The postulates for Horn revision are obtained by recasting the AGM revision postulates to Horn logic.

($H * 1$) $H * \phi = Cn_{\mathrm{H}}(H * \phi)$
($H * 2$) $H * \phi \subseteq H + \phi$
($H * 3$) If $\perp \notin H + \phi$, then $H + \phi \subseteq H * \phi$
($H * 4$) $\phi \in H * \phi$
($H * 5$) If ϕ is consistent, then $\perp \notin H * \phi$
($H * 6$) If $\phi \equiv \psi$, then $H * \phi = H * \psi$
($H * 7$) $H * \phi \wedge \psi \subseteq (H * \phi) + \psi$
($H * 8$) If $\perp \notin (H * \phi) + \psi$ then, $(H * \phi) + \psi \subseteq H * \phi \wedge \psi$

Delgrande and Peppas notice that MHR functions do not in general satisfy ($H * 7$) and ($H * 8$). They provide the following counterexample.

Example 1 [9] Let $\mathcal{P} = \{p, q, r\}$. The agent's belief set H is given by $Cn_{\mathrm{H}}(p \wedge q \wedge r)$ which is assigned the pre-order $\preceq$ such that

$$111 \prec 010 \simeq 100 \prec 001 \prec 000 \prec \text{all other interpretations}$$

Let ϕ be $\neg p \vee \neg q$ and ψ be $\neg p \wedge \neg q$. Thus $|\phi| = \{101, 100, 011, 010, 001, 000\}$, $|\psi| = \{001, 000\}$, and $|\phi \wedge \psi| = \{001, 000\}$. Suppose $*$ is the MHR function for H determined by $\preceq$, then

$$\begin{aligned} |H * \phi| &= Cl_{\cap}(min(|\phi|, \preceq)) = Cl_{\cap}(\{100, 010\}) = \{100, 010, 000\} \\ |(H * \phi) + \psi| &= \{100, 010, 000\} \cap \{001, 000\} = \{000\} \\ |H * \phi \wedge \psi| &= Cl_{\cap}(min(|\phi \wedge \psi|, \preceq)) = \{001\} \end{aligned}$$

Thus $(H * \phi) + \psi$ and $H * \phi \wedge \psi$ are not equivalent and violate both ($H * 7$) and ($H * 8$).

In Example 1, the revision by ϕ yields the intermediate model set $\{100, 010\}$ which is not Horn closed as 100 and 010 induce 000. Thus the corresponding Horn belief set also has 000 among its models. The induced model 000 is not a minimal model of ϕ but is included as one of the resulting models. It interferes with the behaviour of the revision so as to violate ($H * 7$) and ($H * 8$).

As shown, the key to the satisfaction of ($H * 7$) and ($H * 8$) is to assure that the resulting models of a revision consist of only the minimal models of the revising formula. This is achieved by requiring the determining pre-orders to be *Horn compliant* [9]. A preorder $\preceq$ is Horn compliant if and only if for every $\phi \in \mathcal{L}_{\mathrm{H}}$,

$min(|\phi|, \preceq) = Cl_{\cap}(min(|\phi|, \preceq))$. It is not hard to see that a pre-order $\preceq$ is Horn compliant if and only if it satisfies the following condition:

(HC): If $\mu \simeq \nu$, then $\mu \cap \nu \preceq \mu$ for all $\mu, \nu \in \Omega$.

Lemma 3 *A pre-order $\preceq$ is Horn compliant iff it satisfies (HC).*

A MHR function whose determining pre-order is Horn compliant satisfies both $(H*7)$ and $(H*8)$. We refer to such functions as *Horn compliant model-based Horn revision* (HCMHR)[5] functions.

Definition 5 [9] A function $*$ is a HCMHR function iff it is a MHR function whose determining pre-order is Horn compliant.

Delgrande and Peppas provided the following representation theorem for HCMHR.

Theorem 1 [9] *A function $*$ is a HCMHR function iff it satisfies $(H*1)$–$(H*8)$ and the following schema:*

$(Acyc)$ *If for $0 \leq i < n$ we have $(H * \mu_{i+1}) + \mu_i \nvdash \bot$, and $(H * \mu_0) + \mu_n \nvdash \bot$, then $(H * \mu_n) + \mu_0 \nvdash \bot$.*

In addition to $(H*1)$–$(H*8)$, $(Acyc)$ is required to characterise HCMHR. Pre-orders are by definition acyclic, however, some Horn revision functions generated in the same manner as MHR functions but through cyclic orderings of interpretations also satisfy $(H*1)$–$(H*8)$. $(Acyc)$ rules out such functions by enforcing transitivity on the determining orderings.

5 Defining Horn Contraction Model-Theoretically

The MHR function reviewed in the previous section is the only Horn revision proposed so far. Therefore, to investigate the inter-definability between Horn contraction and Horn revision, we have to look into generating a MHR function from a Horn contraction and generating a Horn contraction function from a MHR function. Since MHR is defined model-theoretically, for such an investigation it is natural to consider a Horn contraction that is also defined model-theoretically. This section is devoted to such a model-theoretic approach in defining Horn contraction.

5.1 Model-Based Horn Contraction

Our construction directly adapts the classic approach thus it starts by obtaining an intermediate model set which consists of models of the original Horn belief set and

[5] For uniformity with the terminology in this paper, we name the Horn revision differently to Delgrande and Peppas [9].

the minimal counter-models of the contracting Horn formula. It then returns the corresponding Horn belief set as the contraction outcome. The minimal counter-models are determined through a faithful total pre-order. The construction is called *model-based Horn contraction* (MHC).

Definition 6 A function $\dot{-}$ is a MHC function for H iff

$$H \dot{-} \phi = \mathcal{T}_{\mathrm{H}}(|H| \cup min(|\neg\phi|, \preceq))$$

for all $\phi \in \mathcal{L}\mathrm{H}$, where $\preceq$ is a faithful total pre-order for H.

Since models of a Horn belief set are Horn closed, we have

$$|H \dot{-} \phi| = |\mathcal{T}_{\mathrm{H}}(|H| \cup min(|\neg\phi|, \preceq))| = Cl_{\cap}(|H| \cup min(|\neg\phi|, \preceq)).$$

Models of the contracted Horn belief set $H \dot{-} \phi$ are referred to as the *resulting models* of the contraction.

As for revision, there is no guarantee that the intermediate model set (i.e., $|H| \cup min(|\neg\phi|, \preceq)$) is Horn closed. In fact, as shown in Example 2 there is no condition on pre-orders that can provide such a guarantee.

Example 2 Let $\mathcal{P} = \{a, b\}$. The agent's belief set H is given by $Cn_{\mathrm{H}}(a \wedge \neg b)$. Let ϕ be $a \vee \neg b$. Then $|H| = \{10\}$ and $|\neg\phi| = \{01\}$. Since $|\neg\phi|$ is a singleton set, no matter which pre-order $\preceq$ is associated with H, we have $min(|\neg\phi|, \preceq) = \{01\}$ which implies $min(|\neg\phi|, \preceq) \cup |H| = \{10, 01\}$. Since 10 and 01 induce 00, $min(|\neg\phi|, \preceq) \cup |H|$ is not Horn closed.

In Example 2, the induced model 00, that is neither a model of H nor a minimal counter-model of ϕ, is included as one of the resulting models. As in the revision case, there is no clear epistemological reason to include such induced models. However, given that the resulting belief set has to be Horn expressible, there is no way we can exclude such models and, at the same time, have a reasonable contraction. For the contraction in Example 2 to be successful, the resulting models must contain 01. Thus to exclude 00 from the resulting models and to keep the resulting models Horn expressible, we must also exclude 10 but this means the resulting belief set is not a subset of the original one. In other words, we end up believing something new when we try to eliminate some old belief (i.e., this violates the Horn version of $(K\dot{-}2)$). Fortunately, unlike the revision case, induced models like 00 do not prevent MHC functions from satisfying the supplementary postulates for Horn contraction.

Theorem 2 *If $\dot{-}$ is a MHC function, then it satisfies*

$(H\dot{-}1)$ $\quad H \dot{-} \phi = Cn_H(H \dot{-} \phi)$
$(H\dot{-}2)$ $\quad H \dot{-} \phi \subseteq H$
$(H\dot{-}3)$ $\quad$ *If $\phi \notin H$, then $H \dot{-} \phi = H$*
$(H\dot{-}4)$ $\quad$ *If $\nvdash \phi$, then $\phi \notin H \dot{-} \phi$*
$(H\dot{-}de)$ $\quad$ *If $\psi \in H \setminus H \dot{-} \phi$, then for all $\chi \in \mathcal{HS}(\phi \vee \psi)$, $\chi \notin H \dot{-} \phi$*
$(H\dot{-}6)$ $\quad$ *If $\phi \equiv \psi$, then $H \dot{-} \phi = H \dot{-} \psi$*

($H\dot{-}7$) $\quad H\dot{-}\phi \cap H\dot{-}\psi \subseteq H\dot{-}\phi \wedge \psi$
($H\dot{-}8$) $\quad$ *If* $\phi \notin H\dot{-}\phi \wedge \psi$, *then* $H\dot{-}\phi \wedge \psi \subseteq H\dot{-}\phi$

Theorem 2 shows that a MHC function satisfies Horn versions of all the AGM contraction postulates except Recovery. Absence of Recovery is not a weakness of MHC. Besides its controversy, satisfaction of Recovery is subject to a property (viz, AGM-compliance) of the underlying logic and Horn logic does not have such property [23, 30]. Alternatively, we rely on ($H\dot{-}de$) to play the role of Recovery. ($H\dot{-}de$) originates from the postulate of *Disjunctive Elimination* [13]:[6]

($K\dot{-}de$) If $\psi \in K$ and $\psi \notin K\dot{-}\phi$, then $\phi \vee \psi \notin K\dot{-}\phi$.

In its contrapositive form

$$\text{If } \psi \in K \text{ and } \phi \vee \psi \in K\dot{-}\phi, \text{ then } \psi \in K\dot{-}\phi.$$

($K\dot{-}de$) is "a condition for a sentence ψ 'to survive' the contraction process" [13](page 745). So essentially ($K\dot{-}de$) specifies what should be retained after the contraction and in turn captures some minimal change properties of a contraction. ($K\dot{-}de$) is equivalent to Recovery under the AGM settting but, unlike Recovery, it is amenable to Horn logic. ($H\dot{-}de$) is obtained from ($K\dot{-}de$) by replacing the possibly non-Horn disjunction $\phi \vee \psi$ with its Horn strengthenings.

The postulates in Theorem 2 cannot fully characterise MHC. The reason is that ($H\dot{-}7$) is, in a sense, weaker than its classic counterpart ($K\dot{-}7$), which is essential in characterising model-based contraction. In its model-theoretic form

$$|K\dot{-}\phi \wedge \psi| \subseteq |K\dot{-}\phi| \cup |K\dot{-}\psi|$$

($K\dot{-}7$) requires that the resulting model set of the contraction by $\phi \wedge \psi$ is a subset of the union of the resulting models of the contraction by ϕ and that by ψ. Note that ($H\dot{-}7$) is, in fact, the syntactic Horn adaptation of ($K\dot{-}7$). We can also obtain a model-theoretic Horn adaptation of ($K\dot{-}7$) from its model-theoretic form, thus the following postulate:

($H\dot{-}7m$) $|H\dot{-}\phi \wedge \psi| \subseteq |H\dot{-}\phi| \cup |H\dot{-}\psi|$

Consider the model-theoretic form for ($H\dot{-}7$). Since Lemma 1 implies $|H\dot{-}\phi \cap H\dot{-}\phi| = Cl_\cap(|H\dot{-}\phi| \cup |H\dot{-}\phi|)$, we have

$$|H\dot{-}\phi \wedge \psi| \subseteq Cl_\cap(|H\dot{-}\phi| \cup |H\dot{-}\phi|)$$

which requires that the resulting model set of the contraction by $\phi \wedge \psi$ is a subset of the Horn closure of the union of the resulting models of the contraction by ϕ and that by ψ. It is clear from the model-theoretic form of ($H\dot{-}7$) that it is weaker than ($H\dot{-}7m$). The change mechanism for MHC is the same as its classic counterpart

[6] Disjunctive Elimination was originally proposed in the context of belief base change where the the agent's set of beliefs are not necessarily logically closed. It is adapted to belief set change here where the set of beliefs are assumed to be logically closed.

thus, in analogy to the classic case, we will need $(H\dot{-}7m)$ to characterise MHC. Yet, MHC does not satisfy $(H\dot{-}7m)$, as shown in the next example.

Example 3 Let $\mathcal{P} = \{a, b\}$. The agent's belief set H is given by $Cn_{\mathrm{H}}(a \wedge b)$ which is assigned the pre-order $\preceq$ such that:

$$11 \prec 01 \simeq 10 \prec 00.$$

Let ϕ be a and ψ be b. Thus $|\neg\phi| = \{01, 00\}$, $|\neg\psi| = \{10, 00\}$, and $|\neg\phi| \cup |\neg\psi| = \{01, 00, 10\}$. Suppose $\dot{-}$ is the MHC function determined by $\preceq$, then:

$$\begin{aligned}
&|H\dot{-}\phi| = Cl_{\cap}(|H| \cup min(|\neg\phi|, \preceq)) = Cl_{\cap}(\{11, 01\}) = \{11, 01\}\\
&|H\dot{-}\psi| = Cl_{\cap}(|H| \cup min(|\neg\psi|, \preceq)) = Cl_{\cap}(\{11, 10\}) = \{11, 10\}\\
&|H\dot{-}\phi| \cup |H\dot{-}\phi| = \{11, 01, 10\}\\
&|H\dot{-}\phi \wedge \psi| = Cl_{\cap}(|H| \cup min(|\neg\phi| \cup |\neg\psi|, \preceq)) = Cl_{\cap}(\{11, 10, 01\})\\
&= \{11, 10, 01, 00\}
\end{aligned}$$

Since $|H\dot{-}\phi| \cup |H\dot{-}\psi|$ is a proper subset of $|H\dot{-}\phi \wedge \psi|$, $(H\dot{-}7m)$ is violated.

Due to the discrepancy between the syntactic and model-theoretic adaptations of $(K\dot{-}7)$ to Horn logic, we are unable to identify a characterising set of postulates for MHC.

5.2 Connections with Other Contractions

Although proving a representation result for MHC is difficult, we can reveal its properties by showing its connections with AGM contraction and other Horn contractions. We start by looking at the connection with AGM model-based contraction. It is clear from the definition of MHC that its change mechanism is the same as that of AGM model-based contraction. In fact, considering contractions by Horn formulas only, for each AGM model-based contraction function $-$, there is a MHC function $\dot{-}$ that performs identically with $-$ in terms of the Horn formulas returned. The converse also holds.

Theorem 3 *Let H be a Horn belief set and K a belief set such that $K = Cn(H)$. If $-$ is a model-based contraction function for K, then there is a MHC function $\dot{-}$ for H such that $H\dot{-}\phi = \mathcal{H}(K - \phi)$ for all $\phi \in \mathcal{L}_H$.*

If $\dot{-}$ is a MHC function for H, then there is a model-based contraction function $-$ for K such that $H\dot{-}\phi = \mathcal{H}(K - \phi)$ for all $\phi \in \mathcal{L}_H$.

Theorem 3 assures that MHC performs identically to AGM contraction when restricted to Horn formulas. Now let's proceed to the connections with Horn contractions.

For the existing Horn contractions, *transitively relational partial meet Horn contraction* (TRPMHC) [34] and *entrenchment-based Horn contraction* (EHC) [32, 36] also assume certain forms of plausibility ranking (i.e., relations over weak remainder sets and those over Horn formulas) and satisfy the supplementary postulates for contraction. TRPMHC [33, 34][7] is the Horn adaptation of *transitively relational partial meet contraction* [2]. TRPMHC is based on the notion of *weak remainder sets* [11]. The set of weak remainder sets of H with respect to ϕ, denoted by $H \downarrow_w \phi$, is such that $X \in H \downarrow_w \phi$ if and only if $X = Cn_{\mathrm{H}}(X)$ and $|X| = Cl_{\cap}(|H| \cup \{\mu\})$ for some $\mu \in |\neg\phi|$. A selection function γ for H is such that $\gamma(H \downarrow_w \phi)$ returns a non-empty subset of $H \downarrow_w \phi$ whenever $H \downarrow_w \phi$ is non-empty and returns H otherwise. γ is *transitively relational* if and only if a transitive relation $\leqslant$ over weak remainder sets of H is used to generate γ via the *marking off* identity:

$$\gamma(H \downarrow_w \phi) = \{X \in H \downarrow_w \phi \mid Y \leqslant X \text{ for all } Y \in H \downarrow_w \phi\}.$$

A TRPMHC function $\dot{-}$ is defined as $H \dot{-} \phi = \bigcap \gamma(H \downarrow_w \phi)$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$, where γ is a transitively relational selection function for H. Clearly, the function $\dot{-}$ is determined by the selection function γ which is in turn determined by the relation $\leqslant$.

To see the connection between TRPMHC and MHC, note that each weak remainder set of H with respect to ϕ is determined by a unique counter-model of ϕ. In other words, there is a bijection between elements of $H \downarrow_w \phi$ and those of $|\neg\phi|$. Thus it is no surprise that a relation over weak remainder sets and one over interpretations can be derived from one another. In fact, a faithful pre-order $\preceq$ over Ω for a Horn belief set H can be derived from a transitive relation $\leqslant$ over weak remainder sets for H and vice versa.[8] Moreover, the MHC function determined by $\preceq$ and the TRPMHC function determined by $\leqslant$ perform identically.

Theorem 4 *A function $\dot{-}$ is a MHC function for H iff it is a TRPMHC function for H.*

Theorem 4 assures that MHC is equivalent to TRPMHC. Next, we show the connection with EHC which is not as simple as the one with TRPMHC.

EHC is the Horn adaptation of entrenchment-based contraction. It assumes a binary relation $\leq$ over $\mathcal{L}_{\mathrm{H}}$ called *Horn epistemic entrenchment* which satisfies the following conditions:

$(HEE1)$ If $\phi \leq \psi$ and $\psi \leq \chi$, then $\phi \leq \chi$
$(HEE2)$ If $\phi \vdash \psi$, then $\phi \leq \psi$
$(HEE3)$ $\phi \leq \phi \wedge \psi$ or $\psi \leq \phi \wedge \psi$
$(HEE4)$ If $H \nvdash \bot$, then $\phi \notin H$ iff $\phi \leq \psi$ for every ψ
$(HEE5)$ If $\phi \leq \psi$ for every ϕ, then $\vdash \psi$

$(HEE1)$–$(HEE5)$ are the Horn versions of the conditions $(EE1)$–$(EE5)$ in [16]. As in the classic case, $\phi \leq \psi$ means ψ is at least as entrenched as ϕ and $\phi < \psi$ means

[7] The representation theorem for TRPMHC is given by Zhang and Pagnucco [34]. TRPMHC can be characterised by $(H\dot{-}1)$–$(H\dot{-}4)$, $(H\dot{-}de)$, $(H\dot{-}6)$, $(H\dot{-}pa)$, and $(H\dot{-}8)$. It was subsequently determined that the proof for this theorem was not valid as it contains a subtle error. The correct part of the proof assures that a TRPMHC function satisfies $(H\dot{-}1)$–$(H\dot{-}4)$, $(H\dot{-}de)$, $(H\dot{-}6)$, $(H\dot{-}pa)$, and $(H\dot{-}8)$.

[8] The simple derivation methods can be found in the proof of Theorem 4.

ψ is strictly more entrenched than ϕ. [9] An entrenchment-based contraction function is determined by the associated epistemic entrenchment via the $(C\dot{-})$ condition such that a formula ψ is retained while contracting ϕ if and only if ψ is in the original belief set and, either ϕ is a tautology, or $\phi \vee \psi$ is strictly more entrenched than ϕ [15, 16]. Since the disjunction $\phi \vee \psi$ may be non-Horn, EHC is defined through the adapted condition $(HC\dot{-})$ that replaces the disjunction with its Horn strengthenings.

$(HC\dot{-})$: $\psi \in H\dot{-}\phi$ iff $\psi \in H$ and either $\vdash \phi$ or there is $\chi \in \mathcal{HS}(\phi \vee \psi)$ such that $\phi < \chi$.

Thus an EHC function which is determined by the associated Horn epistemic entrenchment via $(HC\dot{-})$ is such that ψ is retained after the contraction by ϕ if and only if ψ is in the original Horn belief set and, either ϕ is a tautology, or a Horn strengthening of $\phi \vee \psi$ is strictly more entrenched than ϕ.

It has been shown that an EHC function is a TRPMHC function but the converse does not hold [36] in general. Thus the equivalence between TRPMHC and MHC implies that there are MHC functions that are not EHC functions. In other words, MHC is more general than EHC. So our focus is to identify the group of MHC functions that are also EHC functions.

With this in mind, we define the following condition of *strict Horn compliance* on pre-orders:

(SHC): $\mu \cap \nu \preceq \mu$ or $\mu \cap \nu \preceq \nu$ for all $\mu, \nu \in \Omega$.

A pre-order is strict Horn compliant if any interpretation is equally or more preferred than at least one of its inducing interpretations. Strict Horn compliance is clearly a stricter condition than Horn compliance, thus the naming.

Lemma 4 *Let $\preceq$ be a pre-order. If $\preceq$ is strictly Horn compliant, then it is Horn compliant.*

Another obvious result is that, if ω is induced by some elements of a set of interpretations M, then there exists μ of M such that $\omega \preceq \mu$, provided that $\preceq$ is strict Horn compliant.

Lemma 5 *Let $\preceq$ be a strict Horn compliant pre-order and M a set of interpretations. If $\omega \in Cl_\cap(M) \setminus M$, then there is $\mu \in M$ such that $\omega \preceq \mu$.*

A MHC function is a *strict Horn compliant model-based Horn contraction* (SHCMHC) function if its determining pre-order is strict Horn compliant.

Definition 7 A function $\dot{-}$ is a SHCMHC function iff it is a MHC function whose determining pre-order is strict Horn compliant.

We can show that all SHCMHC functions are EHC functions.

[9] Note that for a pre-order of interpretations the minimally ordered interpretations are most preferred whereas for an (Horn) epistemic entrenchment the maximally ordered formulas are most preferred.

Theorem 5 *If a function $\dot{-}$ is a SHCMHC function for H, then it is an EHC function for H.*

As illustrated in the following example, the converse does not hold in general.

Example 4 Let $\mathcal{P} = \{a, b\}$. The agent's belief set H is given by $Cn_{\mathrm{H}}(\neg a \wedge b)$. Suppose the epistemic entrenchment $\leq$ for H is such that

$$\neg a = b = \neg a \vee b < \neg a \vee \neg b$$

and the pre-order $\preceq$ for H is such that

$$01 \prec 10 \simeq 11 \prec 00.$$

Let $\dot{-}_{EHC}$ be the EHC function for H determined by $\leq$ and $\dot{-}_{MHC}$ the MHC function for H determined by $\preceq$. Since $\preceq$ is not strictly Horn compliant, $\dot{-}_{MHC}$ is not a SHCMHC function. Yet it can be shown that $H\dot{-}_{EHC}\phi = H\dot{-}_{MHC}\phi$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$ which means the two functions are identical.

This result completes our quest for a model-theoretic definition of Horn contraction. In the next two sections, we investigate the inter-definability of MHR and MHC. It will be clear that the condition of strict Horn compliance plays a crucial role for defining MHR through MHC.

6 Defining Horn Revision Through Horn Contraction

In this section, we provide methods for generating plausible Horn revision functions from MHC functions. To be precise, we consider a Horn revision function plausible only if it satisfies Horn versions of the full set of AGM revision postulates.

6.1 Dealing with Non-Horn Negations

An obvious obstacle in generating Horn revision and contraction from one another is the lack of negation in Horn logic. Following the Levi identity, a Horn revision function $*$ for H would be derived from a MHC function $\dot{-}$ for H as follows

$$H * \phi = (H\dot{-}\neg\phi) + \phi$$

for all $\phi \in \mathcal{L}_{\mathrm{H}}$. For example, in the revision by $\neg p \wedge \neg q$, we would first contract by the negation of $\neg p \wedge \neg q$, however, the negation (i.e., $p \vee q$) is not a Horn formula which cannot be an input to the MHC function $\dot{-}$.

One natural way to deal with non-Horn negation is to contract by its Horn approximations. The notion of Horn strengthening has proved to be useful in such situations, for instance it is used as the Horn approximations for non-Horn formulas in defining EHC.

A non-Horn formula has at least two Horn strengthenings and each implies the non-Horn formula. To guarantee consistency after the expansion, in the contraction step all Horn strengthenings of the non-Horn negation have to be removed.

The standard approach to remove several items of information simultaneously is to apply a *package contraction* [14]. AGM contraction and Horn contractions studied so far are in fact *singleton contractions* which take a single formula as input and return as output a belief set that does not imply the formula. In contrast, package contraction takes as input a set of formulas and returns as output a belief set that does not imply any formula in the set.

We will not take the package contraction approach as our goal in this paper is to investigate the inter-definability of singleton Horn revision and singleton Horn contraction. Note that package contraction has not been fully explored even in the classic setting. No postulates have been shown to play the same role as the AGM supplementary postulates in characterising package contractions [19]. The identification of such supplementary postulates and the study of package contraction in the Horn setting are beyond the scope of this paper and we plan to tackle them in a future work.

Actually, it is not a desideratum to remove the Horn strengthenings simultaneously. Apart from the method of package contraction, their removal can also be achieved by either performing a sequence of contractions that remove the Horn strengthenings one by one (i.e., in the form of iterated contraction) or performing a contraction for each Horn strengthening and then intersecting the contraction results. Starting with the intersection approach we will show that both approaches give us what we require and are, in fact, equivalent.

6.2 The Intersection Approach

In the *intersection approach*, the revision by ϕ is achieved by first intersecting the outcomes of contractions by each Horn strengthening of $\neg\phi$, then expanding the intersection by ϕ. If the contraction function used in the contraction step is a MHC function, then the revision function thus generated is called an *intersection revision* (IR) function.

Definition 8 A function $*$ is an IR function for H iff

$$H * \phi = [(H \dot{-} \chi_1) \cap (H \dot{-} \chi_2) \cap \cdots \cap (H \dot{-} \chi_n)] + \phi$$

for all $\phi \in \mathcal{L}_{\mathrm{H}}$, where $\dot{-}$ is a MHC function for H and $\{\chi_1, \ldots, \chi_n\} = \mathcal{HS}(\neg\phi)$.

Unfortunately, IR leads to counterintuitive revision outcomes. Consider the Horn language with $\mathcal{P} = \{a, b\}$. Let H and ϕ be such that $|H| = \{01\}$ and $|\phi| = \{00, 11, 10\}$ and the pre-order $\preceq$ for H be $01 \prec 10 \prec 11 \prec 00$. For the revision of H by ϕ, Fig. 1 illustrates, in a system of spheres setting, the resulting models (indicated by the shaded area) of the revision through an IR function (left diagram) and those of the revision through a MHR function (right diagram). According to the intuition of model-based revision, the revision by ϕ should end up with models of ϕ that are closest to those of H (i.e., the minimal models of ϕ by means of the associated pre-order). Contrary to intuition, the IR function in Fig. 1 returns 10 and 00 as the resulting models. Given that 11, which is closer to $|H|$ than 00, is not taken as a resulting model, it does not make any sense to take 00 as a resulting model. The example also suffices to show that IR does not coincide with MHR.

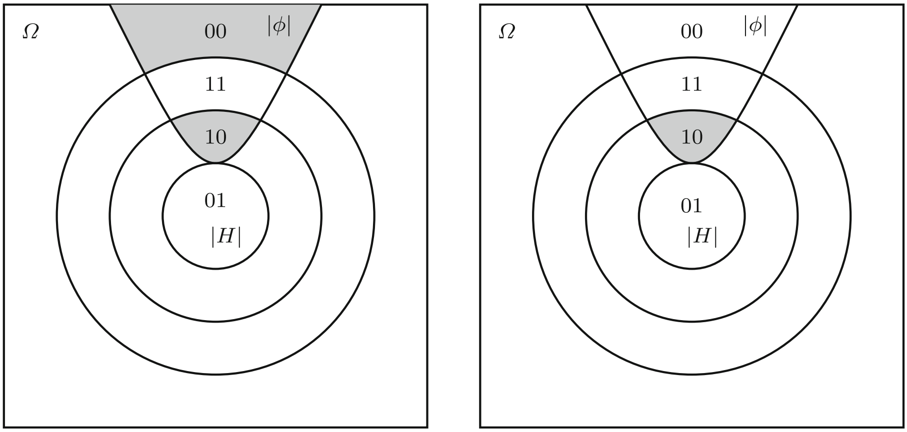

Fig. 1 Resulting models for the revision by ϕ through an IR function (*left diagram*) and a MHR function (*right diagram*)

Informally, the culprit for the abnormality of the IR function in Fig. 1 is the model 00. Since $|\neg\phi| = \{01\} = Cl_\cap(|\neg\phi|)$, $\neg\phi$ is a Horn formula which means its only Horn strengthening is itself. Suppose $*$ is the IR function in Fig. 1 and $\dot{-}$ the MHC function that generates it, then we have

$$|H * \phi| = |(H \dot{-} \neg\phi) + \phi| = Cl_\cap(|H| \cup min(|\phi|, \preceq)) \cap |\phi|.$$

00 is included as a resulting model merely because it is induced by 01 of $|H|$ and 10 of $min(|\phi|, \preceq)$ when applying the Horn closure operator. In other words, there is no epistemological reason for its inclusion. Moreover, recall that the key for MHR functions to satisfy $(H * 7)$ and $(H * 8)$ is to assure their resulting models coincide with the minimal models of the revising formulas. The induced model 00 is not a minimal model of the revising formula ϕ thus it also gives rise to problems with the satisfaction of $(H * 7)$ and $(H * 8)$.

Inclusion of induced models with no epistemological reasons can be avoided if we restrict our attention to pre-orders where induced models are always more preferred than one of their inducing models. Obviously such pre-orders are strict Horn compliant and the Horn contraction functions determined by the pre-orders are SHCMHC functions. We refer to an IR function whose generating Horn contraction function is a SHCMHC function as a *strict Horn compliant intersection revision* (SHCIR) function.

Definition 9 A function $*$ is a SHCIR function iff it is an IR function generated by a SHCMHC function.

The behaviour of SHCIR is in line with the intuition of AGM revision. Specifically, resulting models of a SHCIR function always coincide with minimal models of the revising formula.

Theorem 6 *Let $*$ be a SHCIR function for H such that it is generated by a SHCMHC function with the determining pre-order $\preceq$. Then*

$$|H * \phi| = min(|\phi|, \preceq)$$

for all $\phi \in \mathcal{L}_H$.

It follows immediately from Theorem and the definition of MHR that a SHCIR function is a MHR function. More precisely, a SHCIR function is a MHR function whose determining pre-order is strict Horn compliant. We refer to such MHR functions as *strict Horn compliant model-based Horn revision* (SHCMHR) functions.

Definition 10 A function $*$ is a SHCMHR function iff it is a MHR function whose determining pre-order is strict Horn compliant.

Theorem 7 *A function $*$ is a SHCIR function for H iff it is a SHCMHR function for H.*

Theorem 7 assures that the revision function generated from a SHCMHC function is a SHCMHR function and every SHCMHR function can be generated from a SHCMHC function.

Although an IR function is generated from a contraction function, the generation which involves several contractions followed by an intersection is different from the generation of AGM revision from AGM contraction which involves only one contraction. In fact, the generation process can be reduced to a single contraction provided that the determining pre-order is strict Horn compliant.

Theorem 8 *Let $*$ be a SHCIR function for H such that it is generated by a SHCMHC function $\dot{-}$ with the determining pre-order $\preceq$. Then there is $\chi \in \mathcal{HS}(\neg\phi)$ such that $min(|\neg\chi|, \preceq) \cap |\phi| \neq \emptyset$ and*

$$H * \phi = (H \dot{-} \chi) + \phi$$

for all $\phi \in \mathcal{L}_H$.

According to Theorem 8, each SHCIR function can be generated from a SHCMHC function such that the contraction step involves only the contraction by a single Horn strengthening of the negation of the revising formula.

6.3 The Sequential Approach

Another way to generate Horn revision from Horn contraction is to apply a sequence of singleton contractions followed by an expansion. In this approach, the revision by ϕ is achieved through contracting the Horn strengthenings of $\neg\phi$ one by one and then expanding the final contraction outcome by ϕ. Formally, a Horn revision function $*$ is generated as follows:

$$H * \phi = ((\cdots((H \dot{-}_1 \chi_1) \dot{-}_2 \chi_2) \cdots) \dot{-}_n \chi_n) + \phi$$

where $\{\chi_1, \ldots, \chi_n\} = \mathcal{HS}(\neg\phi)$ and $\dot{-}_1, \ldots, \dot{-}_n$ are Horn contraction functions for H, $H\dot{-}_1\chi_1, \ldots,$ and $((\cdots((H\dot{-}_1\chi_1)\dot{-}_2\chi_2)\cdots)\dot{-}_{n-1}\chi_{n-1})$ respectively.

The need to perform a sequence of contractions gives rise to another difficulty. Singleton contractions are "one shot" operations that do not specify the posterior preference information associated with the contracted belief set, therefore subsequent contractions are not possible. For instance, the MHC function $\dot{-}_1$ does not specify the posterior pre-order for the contracted Horn belief set $H\dot{-}_1\phi$ thus no further contraction can be determined for $H\dot{-}_1\phi$. To overcome this difficulty, we take the standard approach by applying an iteration scheme [7, 29]. Let $-$ be a model-based contraction for K determined by the pre-order $\preceq$. In contracting ϕ, an iteration scheme specifies a posterior pre-order, written as $\preceq^-_\phi$, for the posterior belief set $K - \phi$. In this way, a contraction for the belief set $K - \phi$ can be determined by $\preceq^-_\phi$.

Several iteration schemes are proposed for contraction such as *priority contraction* [28], *conservative contraction* [31], and *lexicographic contraction* [27]. Roughly speaking, priority contraction gives precedence to new beliefs, conservative contraction gives precedence to old beliefs, and lexicographic contraction treats the old and new beliefs the same.

In the current context, the sequence of contractions is intended to replace a single contraction, thus it makes sense to retain as much as possible of the prior pre-order in each iteration so that the sequence of contractions better mimics the behaviour of a single contraction. For this reason, conservative contraction is most suitable as it best retains the prior pre-order. Conservative contraction is characterised by the following conditions.

C1: If $\omega_1 \in min(\Omega, \preceq) \cup min(|\neg\phi|, \preceq)$, then $\omega_1 \preceq^-_\phi \omega_2$ for all $\omega_2 \in \Omega$.

C2: If $\omega_1, \omega_2 \notin min(\Omega, \preceq) \cup min(|\neg\phi|, \preceq)$, then $\omega_1 \preceq^-_\phi \omega_2$ iff $\omega_1 \preceq \omega_2$.

In contracting by ϕ, C1 assures that the resulting models are most preferred in the posterior pre-order which guarantees its faithfulness, and C2 assures that the rest of the posterior pre-order is identical to the prior one.

C1 and C2 have to be modified for iterated Horn contraction. Suppose $-$ is a MHC function for H determined by $\preceq$. There may be $\omega \in |H - \phi|$ such that $\omega \notin |H| \cup min(|\neg\phi|, \preceq)$. Since C2 does not guarantee the minimality of ω in $\preceq^-_\phi$, we have $|H - \phi| \neq min(\Omega, \preceq^-_\phi)$, which means $\preceq^-_\phi$ is not faithful (with respect to $H - \phi$). As the determining pre-orders for MHC functions have to be faithful, $\preceq^-_\phi$ cannot be used for determining MHC functions for $H - \phi$. We therefore modify C1 so that induced models like ω are made minimal and, to avoid conflict, we modify C2 so that it does not apply to such models:

HC1: If $\omega_1 \in Cl_\cap(min(\Omega, \preceq) \cup min(|\neg\phi|, \preceq))$, then $\omega_1 \preceq^-_\phi \omega_2$ for all $\omega_2 \in \Omega$.

HC2: If $\omega_1, \omega_2 \notin Cl_\cap(min(\Omega, \preceq) \cup min(|\neg\phi|, \preceq))$, then $\omega_1 \preceq^-_\phi \omega_2$ iff $\omega_1 \preceq \omega_2$.

In subsequent sections, with any contraction sequence $((\cdots((H -_1 \chi_1) -_2 \chi_2)\cdots) -_n \chi_n)$ we assume that HC1 and HC2 are applied throughout the sequence and we take the convention that $\preceq^{-i}_{\phi_i}$ is the pre-order specified by HC1 and HC2 after contracting ϕ_i through $-_i$ (for $1 \leq i \leq n$) such that $-_1$ is a MHC function determined by the pre-order $\preceq$ for H, and $-_i$ ($2 \leq i \leq n$) is a MHC function determined by the pre-order $\preceq_{\phi_{i-1}}^{-_{i-1}}$ for $((\cdots((H -_1 \phi_1) -_2 \phi_2)\cdots) -_{i-1} \phi_{i-1})$.

In generating a Horn revision function through the sequential approach, if the contraction functions used in the contraction step are MHC functions, then the generated revision function is called a *sequential revision* (SR) function.

Definition 11 A function $*$ is a SR function for H iff

$$H * \phi = ((\cdots((H \dot{-}_1 \chi_1) \dot{-}_2 \chi_2)\cdots) \dot{-}_n \chi_n) + \phi$$

for all $\phi \in \mathcal{L}_{\mathrm{H}}$, where $\dot{-}_1$ is a MHC function for H and $\{\chi_1, \ldots, \chi_n\} = \mathcal{HS}(\neg\phi)$.

Note that once the first contraction function $\dot{-}_1$ is fixed, subsequent contractions are also fixed as their determining pre-orders are specified by HC1 and HC2 from the pre-order for H (which determines $\dot{-}_1$). The SR function $*$ for H is therefore fully determined by the contraction function $\dot{-}_1$ which is referred to as the generating contraction function for $*$. In Definition we did not specify the order in which the Horn strengthenings are contracted. The reason will be discussed towards the end of this section.

Commonly, the belief revision community understands a revision, through the Levi identity, as a contraction followed by an expansion. However, this is a simplification of Levi's original idea [24] as expressed by his *Commensurability Thesis* (page 65) which essentially states that one can get from one state of belief to another through a sequence of expansions and contractions. Therefore, to obtain a revision by performing a sequence of contractions and an expansion as for SR is in accordance with Levi's original idea on the nature of the revision operation.

A SR function without any restriction is problematic. It produces the same counterintuitive revision outcome when applied to the example shown in Fig. 1. Again the problem can be avoided by restricting to strict Horn compliant pre-orders and thus SHCMHC functions. We refer to a SR function whose generating Horn contraction function is a SHCMHC function as a *strict Horn compliant sequential revision* (SHCSR) function.

Definition 12 A function $*$ is a SHCSR function iff it is a SR function generated by a SHCMHC function.

A MHR function is determined by a pre-order over all interpretations, however, the result of a particular revision is determined by the ordering between models of the revising formula. For a SHCSR function, the ordering between models of the revising formula is not altered throughout its contraction sequence.

Lemma 6 *Let $\preceq$ be a pre-order for H that is strict Horn compliant and $\phi \in \mathcal{L}_H$ be such that $\mathcal{HS}(\neg\phi) = \{\chi_1, \chi_2, \ldots, \chi_n\}$. For the contraction sequence $((\cdots((H -_1 \chi_1) -_2 \chi_2)\cdots) -_n \chi_n)$, if $\mu, \nu \in |\phi|$, then*

$$\mu \preceq \nu \text{ iff } \mu \preceq_{\chi_i}^{-_i} \nu$$

for $1 \leq i \leq n$.

With Lemma 6 we can show that resulting models of a SHCSR function always coincide with minimal models of the revising formula.

Theorem 9 *Let* $*$ *be a SHCSR function for* H *such that it is generated by a SHCMHC function with the determining pre-order* $\preceq$. *Then*

$$|H * \phi| = min(|\phi|, \preceq)$$

for all $\phi \in \mathcal{L}_H$.

The behaviour is identical to that of SHCIR. Clearly, SHCIR and SHCSR are equivalent.

Theorem 10 *A function* $*$ *is a SHCIR function iff it is a SHCSR function.*

Due to the equivalence between SHCIR and SHCSR, it follows from Theorem 8 that SHCSR can also be reduced to a single contraction followed by an expansion. Although the contraction part of two SHCSR functions may differ in the ordering, the two contraction sequences are reducible to a contraction by an identical formula. Thus the ordering in which the contractions are performed does not affect the result of SHCSR functions; an initial SHCMHC and an iteration scheme are all we need. This is why we did not specify the ordering of contractions for SR functions.

7 Defining Horn Contraction Through Horn Revision

In this section, we focus on the generation of Horn contraction from Horn revision. In particular, we provide a method for generating plausible Horn contraction functions from MHR functions. We consider a Horn contraction function plausible only if it satisfies $(H\dot{-}1)$–$(H\dot{-}4)$, $(H\dot{-}de)$, and $(H\dot{-}6)$–$(H\dot{-}8)$.

According to Harper [20], such generation consists of a revision step followed by an intersection step. In the revision step we need to revise by the negation of the contracting formula. Again, the negation may be non-Horn. Inspired by the intersection approach for generating revision from contraction, we can first obtain their Horn strengthenings, then revise them separately and finally take the intersection of the revision outcomes. Formally, a contraction function $\dot{-}$ is generated from a revision function $*$ as follows:

$$H\dot{-}\phi = (H * \chi_1) \cap (H * \chi_2) \cap \cdots \cap (H * \chi_n) \cap H$$

for all $\phi \in \mathcal{L}_{\mathrm{H}}$, where $\{\chi_1, \ldots, \chi_n\} = \mathcal{HS}(\neg\phi)$. This construction however leads to counterintuitive revision outcomes.

Consider the Horn language with $\mathcal{P} = \{a, b, c\}$. Let $|H| = \{000\}$, the Horn formula ϕ be such that $|\phi| = \{000, 111, 110, 010, 100\}$ and the pre-order $\preceq$ for H as depicted in Fig. 2, in a system of spheres setting. Let's consider the contraction of H by ϕ. If the contraction is performed through the MHC function determined by $\preceq$,

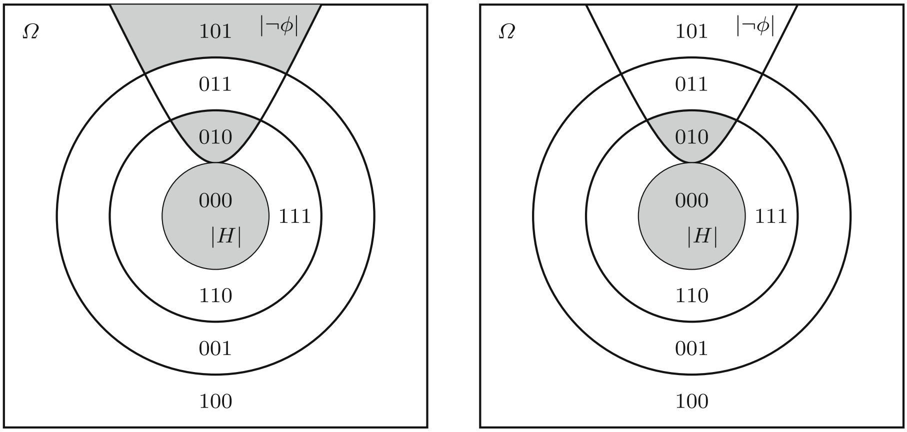

Fig. 2 Resulting models for the contraction by ϕ through a contraction function generated from a MHR using intersection (*left diagram*) and a MHC function (*right diagram*)

the resulting models are obtained by taking the Horn closure of the union of the minimal models of $\neg\phi$ (i.e., 010) and $|H|$. Such models are indicated in the the shaded area of Fig. 2 (right diagram). If the contraction is performed through a contraction function generated as above from a MHR function determined by $\preceq$, then we obtain something different. Since $|\neg\phi| = \{101, 011, 010\} \neq Cl_\cap(\{101, 011, 010\})$, it is non-Horn. So we first obtain its set of Horn strengthenings which consists of χ_1 with $|\chi_1| = \{011, 010\}$ and χ_2 with $|\chi_2| = \{101\}$. The resulting model for the revision by χ_1 is $\{010\}$ and that for the revision by χ_2 is $\{101\}$. Intersecting these models with $|H|$ gives us the resulting models of the contraction which is indicated as the shaded area in Fig. 2 (left diagram).

According to the intuition of model-based contraction, the contraction by ϕ should end up with models of $\neg\phi$ that are closest to those of H (i.e., the minimal models of $\neg\phi$ with respect to a pre-order) and models of H. Contrary to intuition, the generated contraction function returns 000, 010, and 101 as the resulting models. Given that 011, which is closer to $|H|$ than 101, is not taken as a resulting model, it does not make any sense taking 101 as a resulting model. As argued at the end of Section 5, we may tolerate models induced from models that are epistemologically sound (i.e., models 000, 010). Since models 000 and 010 do not induce any new models, there is absolutely no grounds to include model 101 among the resulting models. Also the example suffices to show the generated contraction does not coincide with MHC.

It is not hard to identify model 101 as the culprit for the counterintuitive outcome. In the revision step, each revision returns the minimal model of a Horn strengthening. By the definition of Horn strengthening, the set of models for each Horn strengthening of $\neg\phi$ is a proper subset of $|\neg\phi|$ thus minimal models of $\neg\phi$ are also minimal for some Horn strengthenings, however, minimal models such as the model 101 for some Horn strengthenings are not guaranteed to be minimal for $\neg\phi$. Consequently, the resulting models for the contraction generated as above contains all resulting models for a MHC function but may also contain non-minimal models which cause problems.

Previously, in generating Horn revision from contraction, the intermediate (contraction) step removed all Horn strengthenings of the non-Horn negation to avoid any potential inconsistency. In generating Horn contraction from revision, the intermediate (revision) step is for ceasing the entailment of ϕ and it is sufficient to incorporate one Horn strengthening for this purpose. Thus, one way to deal with the problem is to focus on Horn strengthenings whose minimal models are also minimal with respect to $\neg\phi$ and ignore the rest. For this we need a preference relation over the Horn strengthenings.

We define the *most preferred formulas*[10] of a set X with respect to a pre-order $\preceq$, denoted $min(X, \preceq)$, to be those formulas whose minimal models are minimal among the minimal models of all formulas in X. Formally,
$min(X, \preceq) = \{\phi \in X \mid \text{there is no } \psi \in X \text{ such that } \nu \prec \mu \text{ for } \mu \in min(|\phi|, \preceq) \text{ and } \nu \in min(|\psi|, \preceq)\}$.

If the set of formulas X consists of Horn strengthenings of a formula ϕ, then the union of the minimal models of the most preferred formulas of X coincides with the minimal models of ϕ.

Lemma 7 *Let $\preceq$ be a pre-order. If $min(\mathcal{HS}(\phi), \preceq) = \{\chi_1, \ldots, \chi_n\}$, then*

$$min(|\phi|, \preceq) = min(|\chi_1|, \preceq) \cup \cdots \cup min(|\chi_n|, \preceq).$$

With the intersection approach, if the revision function used in the revision step is a MHR function and only the most preferred Horn strengthenings get revised, then the contraction generated is called an *intersection contraction* (IC) function.

Definition 13 A function $\dot{-}$ is an IC function for H iff

$$H \dot{-} \phi = (H * \chi_1) \cap (H * \chi_2) \cap \cdots \cap (H * \chi_n) \cap H$$

for all $\phi \in \mathcal{L}_{\mathrm{H}}$, where $*$ is a MHR function for H that is determined by a pre-order $\preceq$ and $\{\chi_1, \ldots, \chi_n\} = min(\mathcal{HS}(\neg\phi), \preceq)$.

When applied to the contraction in Fig. 2, the IC function has in the intermediate revision step only the revision by χ_1 as it is the most preferred among Horn strengthenings of $\neg\phi$. Obviously, the IC function returns the same contraction outcome as the MHC function. In fact, we can show that IC is equivalent to MHC.

Theorem 11 *A function $\dot{-}$ is a MHC function for H iff it is an IC function for H.*

Theorem 11 implies that the Horn contraction function generated from a MHR function is a MHC function and each MHC function can be generated from a

[10]Such preference over formulas is also used by Boutilier [6] under the name of *degree of surprise*.

MHR function. Due to the one to one correspondence between MHR functions and MHC functions, the contraction functions generated from HCMHR and SHCMHR correspond to HCMHC and SHCMHC functions respectively.

Recall that MHR is problematic as it violates the supplementary postulates for revision. It is interesting to observe that the problematic revision can actually generate a plausible contraction (i.e., MHC). This observation and the fact that, without restriction on the determining pre-order, MHR is problematic whereas MHC exhibits nice properties even without any restriction on the determining pre-order, lead to the conjecture that it is relatively easier to construct a plausible Horn contraction than Horn revision.

Observe that, unlike the generation of Horn revision, we only presented the intersection approach. The reason is that, in generating Horn contraction, the sequential approach does not work. This is due to the difference between contraction and revision when performed in a sequence. Contraction is cumulative in the sense that the result of a sequence of contractions is obtained by accumulating the results of each single contraction. Revision is overriding in the sense that the result of a sequence of revisions is determined by the last one which overrides all previous results. If we replace the revision step of IC with a sequence of revisions, then the contraction generated will solely depend on the choice of the last Horn strengthening by which it is revised. It is also for this reason that we cannot reduce the generation of Horn contraction to a single revision followed by an intersection.

8 Conclusion

In this paper, we first provided a model-theoretic construction of Horn contraction called MHC. MHC and the MHR given in [9] are direct adaptations of AGM model-based contraction and revision respectively, however, they behave quite differently in terms of the supplementary postulates. While MHC satisfies both supplementary postulates for contraction, MHR violates both of them for revision. We then investigated the inter-definability of MHC and MHR. Due to the generality of MHC and MHR, the inter-definability is representative of Horn contraction and revision in general. Our investigation shows that Horn contraction does not lead to plausible Horn revision and only a restricted class of Horn contractions does so. The situation is quite different for the reverse direction. Horn revision always leads to plausible Horn contraction.

According to the Levi and Harper identities, the intermediate contraction or revision step in generating AGM revision or contraction involves a single contraction or revision operation. With our methods for defining Horn contraction and revision, a noticeable difference from the AGM case is that the intermediate contraction and revision step may involve multiple contraction and revision operations. Despite the difference, our methods succeed in generating plausible Horn contraction and revision. This calls for reconsidering the standard single step inter-generation of contraction and revision. As this has been the focus for AGM contraction and revision, it seems that a generalisation to non-classical logic is also required for the Levi and Harper identities, which will be our first future work.

We have focused on the inter-definability of singleton Horn contraction and revision. This is why we did not use package contraction and revision in the intermediate revision and contraction step of generating Horn revision and contraction functions. Our second future work is to pursue the package contraction and revision approach and identify the necessary properties of a package Horn contraction for generating plausible singleton Horn revision and the dual problem for generating plausible singleton Horn contraction.

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Appendix A: Proofs of Results

Proof for Lemma 1

1. By model theory $H_2 \subseteq H_1$ implies $|H_1| \subseteq |H_2|$. Again by model theory $|H_1| \subseteq |H_2|$ implies $Cn(H_2) \subseteq Cn(H_1)$. Since $H_1 = Cn_{\mathrm{H}}(H_1)$ and $H_2 = Cn_{\mathrm{H}}(H_2)$, $Cn(H_2) \subseteq Cn(H_1)$ implies $H_2 \subseteq H_1$.
2. Immediate from Observation 1.28 in [18].
3. It can be derived from $H_1 = Cn_{\mathrm{H}}(H_1)$ and $H_2 = Cn_{\mathrm{H}}(H_2)$ that $H_1 \cap H_2 = \mathcal{H}(Cn(H_1) \cap Cn(H_2))$. By model theory we have $|Cn(H_1) \cap Cn(H_2)| = |H_1| \cup |H_2|$. Then by the Horn closure property we have $|\mathcal{H}(Cn(H_1) \cap Cn(H_2))| = Cl_{\cap}(|H_1| \cup |H_2|)$, thus $|H_1 \cap H_2| = Cl_{\cap}(|H_1| \cup |H_2|)$.

4. 5. Immediate from the definition of $Cl_{\cap}$. $\square$

Proof for Lemma 2 We first show that if ψ is a Horn formula such that $\psi \vdash \phi$, then there is $\chi \in \mathcal{HS}(\phi)$ such that $\psi \vdash \chi$. If $\psi \in \mathcal{HS}(\phi)$, then the result trivially holds. Suppose $\psi \notin \mathcal{HS}(\phi)$, then by the definition of Horn strengthening there is $\chi_1 \in \mathcal{L}_{\mathrm{H}}$ such that $|\psi| \subset |\chi_1| \subseteq |\phi|$. Again, if $\chi_1 \notin \mathcal{HS}(\phi)$, then there must be $\chi_2 \in \mathcal{L}_{\mathrm{H}}$ such that $|\chi_1| \subset |\chi_2| \subseteq |\phi|$. Since $|\phi|$ is finite, eventually we will find a χ_n which is a Horn strengthening of ϕ. Since $|\psi| \subset |\chi_n|$, $\psi \vdash \chi_n$.

Now we show $|\phi| = |\chi_1| \cup \cdots \cup |\chi_n|$. $|\chi_1| \cup \cdots \cup |\chi_n| \subseteq |\phi|$ follows directly from the definition of Horn strengthening. For the other inclusion, assume there is $\mu \in |\phi|$ such that $\mu \notin |\chi_1| \cup \cdots \cup |\chi_n|$. Let $\psi \in \mathcal{L}$ be such that $|\psi| = \{\mu\}$. Since $Cl_{\cap}(\{\mu\}) = \{\mu\}$, ψ is a Horn formula and $|\psi| \subseteq |\phi|$. Then by the above result there is $\chi \in \mathcal{HS}(\phi)$ such that $|\psi| \subseteq |\chi|$. A contradiction ensues. $\square$

Proof for Theorem 2 Suppose $\dot{-}$ is a MHC function for H with the determining pre-order $\preceq$. $(H\dot{-}1)$, $(H\dot{-}2)$, $(H\dot{-}4)$, and $(H\dot{-}6)$ follow immediately from the construction of MHC. Let's show the proof for the rest of the postulates.

$(H\dot{-}3)$: Suppose $\phi \notin H$. Then we have $|\neg\phi| \cap |H| \neq \emptyset$. Thus by the faithfulness of $\preceq$ we have $min(|\neg\phi|, \preceq) \subseteq |H|$. Thus $|H\dot{-}\phi| = Cl_{\cap}(min(|\neg\phi|, \preceq) \cup |H|) = Cl_{\cap}(|H|) = |H|$. Then $H\dot{-}\phi = H$ follows from $(H\dot{-}1)$.

$(H\dot{-}de)$: We first show that $\dot{-}$ satisfies the following postulate

$$(H\dot{-}mc) \text{ If } \psi \in H \setminus H\dot{-}\phi, \text{ then } |H\dot{-}\phi| \not\subseteq |\phi \vee \psi|.$$

Suppose $\psi \in H \setminus H\dot{-}\phi$, it suffices to show there is $\mu \in |H\dot{-}\phi| \cap |\neg\phi|$ such that $\mu \notin |\psi|$. Assume to the contrary that $|H\dot{-}\phi| \cap |\neg\phi| \subseteq |\psi|$. Then we have, by the construction of MHC, that $Cl_\cap(min(|\neg\phi|, \preceq) \cup |H|) \cap |\neg\phi| \subseteq |\psi|$. It then follows from $min(|\neg\phi|, \preceq) \subseteq |\neg\phi|$ that $min(|\neg\phi|, \preceq) \subseteq |\psi|$. Since $\psi \in H$, we have $|H| \subseteq |\psi|$. Thus $min(|\neg\phi|, \preceq) \cup |H| \subseteq |\psi|$ which implies $Cl_\cap(min(|\neg\phi|, \preceq) \cup |H|) \subseteq |\psi|$. However, it follows from $\psi \notin H\dot{-}\phi$, by the construction of MHC that $Cl_\cap(min(|\neg\phi|, \preceq) \cup |H|) \not\subseteq |\psi|$. So we have a contradiction and this completes the proof of $(H\dot{-}mc)$.

Now suppose $\psi \in H \setminus H\dot{-}\phi$. Then it follows from $(H\dot{-}mc)$ that $|H\dot{-}\phi| \not\subseteq |\phi \vee \psi|$. Thus there is $\mu \in |H\dot{-}\phi|$ such that $\mu \notin |\phi \vee \psi|$. Since $|\chi| \subseteq |\phi \vee \psi|$ for all $\chi \in \mathcal{HS}(\phi \vee \psi)$, $\mu \notin |\chi|$ for all $\chi \in \mathcal{HS}(\phi \vee \psi)$. Then it follows from $\mu \in |H\dot{-}\phi|$ that $H\dot{-}\phi \nvdash \chi$ for all $\chi \in \mathcal{HS}(\phi \vee \psi)$.

$(H\dot{-}7)$: By the definition of MHC, we have

$$|H\dot{-}\phi \wedge \psi| = Cl_\cap(|H| \cup min(|\neg\phi| \cup |\neg\psi|, \preceq))$$

and

$$Cl_\cap(|H\dot{-}\phi| \cup |H\dot{-}\psi|) = Cl_\cap(|H| \cup min(|\neg\phi|, \preceq) \cup min(|\neg\psi|, \preceq)).$$

It is easy to see that $|H| \cup min(|\neg\phi| \cup |\neg\psi|, \preceq) \subseteq |H| \cup min(|\neg\phi|, \preceq) \cup min(|\neg\psi|, \preceq)$. Thus $|H\dot{-}\phi \wedge \psi| \subseteq Cl_\cap(|H\dot{-}\phi| \cup |H\dot{-}\psi|)$ which implies $(H\dot{-}\phi) \cap (H\dot{-}\psi) \subseteq H\dot{-}\phi \wedge \psi$.

$(H\dot{-}8)$: If $\vdash \phi$, $\vdash \psi$, $\phi \notin H$ or $\psi \notin H$, then $(H\dot{-}8)$ is trivially satisfied. So suppose $\phi, \psi \in H$, $\nvdash \phi$, and $\nvdash \psi$. Let $\phi \notin H\dot{-}\phi \wedge \psi$, we need to show $H\dot{-}\phi \wedge \psi \subseteq H\dot{-}\phi$. By the construction of MHC, it suffices to show $Cl_\cap(Min(|\neg\phi|, \preceq) \cup |H|) \subseteq Cl_\cap(min(|\neg\phi| \cup |\neg\psi|, \preceq) \cup |H|)$. Since $\phi \notin H\dot{-}\phi \wedge \psi$, we have $Cl_\cap(min(|\neg\phi| \cup |\neg\psi|, \preceq) \cup |H|) \not\subseteq |\phi|$ which implies $min(|\neg\phi| \cup |\neg\psi|, \preceq) \cup |H| \not\subseteq |\phi|$. Since $\phi \in H$, we have $|H| \subseteq |\phi|$. Thus there is $\mu \in min(|\neg\phi| \cup |\neg\psi|, \preceq)$ such that $\mu \in |\neg\phi|$. Then we have for all $\nu \in |\neg\phi| \cup |\neg\psi|$, $\mu \preceq \nu$. Let $\omega \in min(|\neg\phi|, \preceq)$. It then follows from $\mu \in |\neg\phi|$, that $\omega \preceq \mu$. By the transitivity of $\preceq$, $\omega \preceq \nu$ follows from $\omega \preceq \mu$ and $\mu \preceq \nu$. As $\omega \in |\neg\phi| \cup |\neg\psi|$, we have $\omega \in min(|\neg\phi| \cup |\neg\psi|, \preceq)$. Thus $min(|\neg\phi|, \preceq) \subseteq min(|\neg\phi| \cup |\neg\psi|, \preceq)$ which implies $Cl_\cap(min(|\neg\phi|, \preceq) \cup |H|) \subseteq Cl_\cap(min(|\neg\phi| \cup |\neg\psi|, \preceq) \cup |H|)$. □

Proof for Theorem 3 For the first part, suppose $-$ is a model-based contraction function for K with a determining pre-order $\preceq$. Define $\preceq_{\mathrm{H}}$ as follows:

1. If $\mu, \nu \notin |K| \setminus |H|$, then $\mu \preceq_{\mathrm{H}} \nu$ iff $\mu \preceq \nu$,
2. If $\mu, \nu \in |K| \setminus |H|$, then $\mu \simeq_{\mathrm{H}} \nu$, $\mu \prec_{\mathrm{H}} \omega_1$, and $\omega_2 \prec_{\mathrm{H}} \mu$ for all $\omega_1 \notin |K|$ and $\omega_2 \in |H|$.

Clearly $\preceq_{\mathrm{H}}$ is a faithful pre-order for H. Let $\dot{-}$ be a MHC for H that is determined by $\preceq_{\mathrm{H}}$. It remains to show $H\dot{-}\phi = \mathcal{H}(K - \phi)$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$.

$\supseteq$: Suppose $\psi \in \mathcal{L}_{\mathrm{H}}$ and $\psi \in K - \phi$. We need to show $\psi \in H\dot{-}\phi$. Since $\psi \in K - \phi$, we have by $(K\dot{-}2)$ that $\psi \in K$. Since H contains all the Horn formulas of K, we have $\psi \in H$. If $\phi \notin K$, then since H is a subset of K we have $\phi \notin H$ which implies by $(H\dot{-}3)$ that $H = H\dot{-}\phi$. Thus $\psi \in H\dot{-}\phi$ for the case of $\phi \notin K$. Now suppose $\phi \in K$. Then we have $|\neg\phi| \cap |K| = \emptyset$. Since $\psi \in K - \phi$, we have by the construction of model-based contraction that $min(|\neg\phi|, \preceq) \subseteq |\psi|$. By the definition of $\preceq_{\mathrm{H}}$ (part 1), we have $min(|\neg\phi|, \preceq_{\mathrm{H}}) = min(|\neg\phi|, \preceq)$. Thus $min(|\neg\phi|, \preceq_{\mathrm{H}}) \subseteq |\psi|$. Since $\psi \in H$ implies $|H| \subseteq |\psi|$, we have $min(|\neg\phi|, \preceq_{\mathrm{H}}) \cup |H| \subseteq |\psi|$ which implies $Cl_{\cap}(min(|\neg\phi|, \preceq_{\mathrm{H}}) \cup |H|) \subseteq |\psi|$. Finally, it follows from the construction of MHC, $|H\dot{-}\phi| \subseteq |\psi|$.

$\subseteq$: Suppose $\psi \in H\dot{-}\phi$. We need to show $\psi \in K-\phi$. Since $\psi \in H\dot{-}\phi$ we have by $(H\dot{-}2)$ that $\psi \in H$. Since H is a subset of K, we have $\psi \in K$. If $\phi \notin H$, then since H contains all the Horn formulas of K, we have $\psi \notin K$ which implies by $(K\dot{-}3)$ that $K = K - \phi$. Thus $\psi \in K - \phi$ for the case of $\phi \notin H$. Now suppose $\phi \in H$. Then $\phi \in K$ which implies $|\neg\phi| \cap |K| = \emptyset$. Since $\psi \in H\dot{-}\phi$, we have by the construction of MHC, $Cl_{\cap}(min(|\neg\phi|, \preceq_{\mathrm{H}}) \cup |H|) \subseteq |\psi|$ which implies $min(|\neg\phi|, \preceq_{\mathrm{H}}) \subseteq |\psi|$. By the definition of $\preceq_{\mathrm{H}}$ (part 1), we have $min(|\neg\phi|, \preceq_{\mathrm{H}}) = min(|\neg\phi|, \preceq)$. Thus $min(|\neg\phi|, \preceq) \subseteq |\psi|$. Since $\psi \in K$ implies $|K| \subseteq |\psi|$, we have $min(|\neg\phi|, \preceq) \cup \subseteq |\psi|$ which implies by the construction of model-based contraction that $|K-\phi| \subseteq |\psi|$.

The second part can be proved in a similar manner. This time we need to generate a pre-order for K from the one for H. □

Proof for Theorem 4 It suffices to show (1) if $\dot{-}$ is a MHC function for H, then there is a TRPMHC function $-$ for H such that $H\dot{-}\phi = H - \phi$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$ and (2) if $\dot{-}$ is a TRPMHC function for H, then there is a MHC function $-$ for H such that $H\dot{-}\phi = H - \phi$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$.

For the first part, suppose $\dot{-}$ is a MHC function for H and is determined by the pre-order $\preceq$. We first derive a relation $\leqslant$ over all weak remainder sets of H as follows:

For each pair X, Y of weak remainder sets of H, let $Y \leqslant X$ iff X, Y are such that $|X| = Cl_{\cap}(|H| \cup \{\mu\})$, $|Y| = Cl_{\cap}(|H| \cup \{\nu\})$, and $\mu \preceq \nu$.

Now we show $\leqslant$ is transitive. Suppose X, Y, Z are weak remainder sets of H, $X \leqslant Y$ and $Y \leqslant Z$, it suffices to show $X \leqslant Z$. By the derivation of $\leqslant$, there are interpretations μ, ν, and ω such that $|X| = Cl_{\cap}(|H| \cup \{\mu\})$, $|Y| = Cl_{\cap}(|H| \cup \{\nu\})$, and $|Z| = Cl_{\cap}(|H| \cup \{\omega\})$. Moreover, it follows from $X \leqslant Y$ that $\nu \preceq \mu$ and it follows from $Y \leqslant Z$ that $\omega \preceq \nu$. Since $\preceq$ is transitive, it follows from $\nu \preceq \mu$ and $\omega \preceq \nu$ that $\omega \preceq \mu$. Again by derivation of $\leqslant$, it follows from $\omega \preceq \mu$, that $X \leqslant Z$.

Since $\leqslant$ is a transitive relation over all weak remainder sets of H, it can generate a TRPMHC function for H. Suppose the generated function is $-$, it remains to show $H\dot{-}\phi = H - \phi$ for all $\phi \in \mathcal{L}_H$. If $\phi \notin H$ or $\vdash \phi$, then we can easily obtain $H\dot{-}\phi = H - \phi = H$. So suppose $\phi \in H$ and $\nvdash \phi$. Let $min(|\neg\phi|, \preceq) = \{\mu_1, \ldots, \mu_n\}$. Then we have by the definition of MHC that $H\dot{-}\phi = \mathcal{T}_H(|H| \cup \{\mu_1, \ldots, \mu_n\})$. By the definition of weak remainder set, there are $X_1, \ldots, X_n \in H \downarrow_w \phi$ such that $|X_i| = Cl_{\cap}(|H| \cup \{\mu_i\})$ for $1 \leq i \leq n$. Since $min(|\neg\phi|, \preceq) = \{\mu_1, \ldots, \mu_n\}$, we have, by the derivation of $\leqslant$ that $\{X \in H \downarrow_w \phi \mid Y \leqslant X \text{ for all } Y \in H \downarrow_w \phi\} = \{X_1, \ldots, X_n\}$. Then we have by the definition of TRPMHC that $H - \phi = X_1 \cap \cdots \cap X_n = \mathcal{T}_H(|X_1 \cap \cdots \cap X_n|) = \mathcal{T}_H(Cl_{\cap}(|H| \cup \{\mu_1\} \cup \cdots \cup |H| \cup \{\mu_n\})) = \mathcal{T}_H(Cl_{\cap}(\{\mu_1, \ldots, \mu_n\} \cup |H|)) = H\dot{-}\phi$.

The second part can be proved in a similar manner. This time we need to derive a pre-order over Ω for H from a transitive relation over all weak remainder sets of H. $\square$

Proof for Theorem 5 Let $\dot{-}$ be a SHCMHC function for H. We need to show there is a EHC function $-$ for H such that $H\dot{-}\phi = H - \phi$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$.

Let $\preceq$ be a pre-order. We use $min(|\phi|, \preceq) \preceq min(|\psi|, \preceq)$ to denote that for all $\mu \in min(|\psi|, \preceq)$ there is $\nu \in min(|\phi|, \preceq)$ such that $\nu \preceq \mu$ and $min(|\phi|, \preceq) \prec min(|\psi|, \preceq)$ to denote that for all $\mu \in min(|\psi|, \preceq)$ there is $\nu \in min(|\phi|, \preceq)$ such that $\nu \prec \mu$.

We first show that if a MHC function $\dot{-}$ for H is determined by the pre-order $\preceq$, then $\dot{-}$ satisfies the following condition:

$$\psi \in H\dot{-}\phi iff \psi \in H \text{ and either} \vdash \phi \text{ or } min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq).$$

For one direction, suppose $\psi \in H\dot{-}\phi$ and $\nvdash \phi$. We need to show $\psi \in H$ and $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$. Since $\psi \in H\dot{-}\phi$, $\psi \in H$ follows from $(H\dot{-}2)$. It remains to show $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$. If $|\neg\phi \wedge \neg\psi| = \emptyset$, then the result holds trivially. So suppose $|\neg\phi \wedge \neg\psi| \neq \emptyset$. By the definition of MHC, $\psi \in H\dot{-}\phi$ implies $Cl_\cap(|H| \cup min(|\neg\phi|, \preceq)) \subseteq |\psi|$. Thus $min(|\neg\phi|, \preceq) \cap |\neg\psi| = \emptyset$. Let $\mu \in min(|\neg\phi \wedge \neg\psi|, \preceq)$. Then we have for all $\nu \in min(|\neg\phi|, \preceq)$, $\nu \prec \mu$ for otherwise $min(|\neg\phi|, \preceq) \cap |\neg\psi| \neq \emptyset$. Thus $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$.

For the other direction, suppose $\psi \in H$, $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$, and $\nvdash \phi$. We need to show $\psi \in H\dot{-}\phi$. Since $\psi \in H$, we have $|H| \subseteq |\psi|$. Since $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$, we have $min(|\neg\phi|, \preceq) \cap |\neg\psi| = \emptyset$ which implies $min(|\neg\phi|, \preceq) \subseteq |\psi|$. Thus $Cl_\cap(|H| \cup min(|\neg\phi|, \preceq)) \subseteq |\psi|$ which implies by the definition of MHC that $\psi \in H\dot{-}\phi$.

Now we show that if $\preceq$ is strict Horn compliant, then $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$ iff there is $\chi \in \mathcal{HS}(\phi \vee \psi)$ such that $min(|\neg\phi|, \preceq) \prec min(|\neg\chi|, \preceq)$.

For one direction, suppose $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$. We have to show there is $\chi \in \mathcal{HS}(\phi \vee \psi)$ such that $min(|\neg\phi|, \preceq) \prec min(|\neg\chi|, \preceq)$. If $\phi \vee \psi$ is a Horn formula, then its only Horn strengthening is itself thus the result holds trivially. So suppose $\phi \vee \psi$ is non-Horn. Then there exist two largest (by set inclusion) model sets $X, Y \subseteq |\phi \vee \psi|$ such that $X \cap Y = \emptyset$, for each $x \in X$ there is $y \in Y$ such that $x \cap y = w$ and $w \in |\neg\phi \wedge \neg\psi|$, and for each $y \in Y$ there is $x \in X$ such that $x \cap y = w$ and $w \in |\neg\phi \wedge \neg\psi|$. Since $\preceq$ is strict Horn compliant, for each pair of x and y we have either $w \preceq x$ or $w \preceq y$. Assume, without loss of generality, that $w \preceq y$ for all such pairs. Let $|\chi| = |\phi \vee \psi| \setminus Y$ then, by the definition of Horn strengthening, we have $\chi \in \mathcal{HS}(\phi)$. Since $|\neg\chi| = |\neg\phi \wedge \neg\psi| \cup Y$, then by the derivation of χ we have $min(|\neg\phi \wedge \neg\psi|, \preceq) \preceq min(|\neg\chi|, \preceq)$. Then by the transitivity of $\preceq$ we have $min(|\neg\phi|, \preceq) \prec min(|\neg\chi|, \preceq)$.

For the other direction, suppose there is $\chi \in \mathcal{HS}(\phi \vee \psi)$ such that $min(|\neg\phi|, \preceq) \prec min(|\neg\chi|, \preceq)$. We have to show $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$. Since $|\chi| \subseteq |\phi \vee \psi|$ we have $|\neg\phi \wedge \neg\psi| \subseteq |\neg\chi|$ which implies $min(|\neg\chi|, \preceq) \preceq min(|\neg\phi \wedge \neg\psi|, \preceq)$. Then by the transitivity of $\preceq$ we have $min(|\neg\phi|, \preceq) \prec min(|\neg\phi \wedge \neg\psi|, \preceq)$.

It follows from the above results that if $\dot{-}$ is a SHCMHC function for H that is determined by the pre-order $\preceq$, then it satisfies the following condition:

$(SHC\dot{-})$: $\psi \in H\dot{-}\phi$ iff $\psi \in H$ and either $\vdash \phi$ or there is $\chi \in \mathcal{HS}(\phi \vee \psi)$ such that $min(|\neg\phi|, \preceq) \prec min(|\neg\chi|, \preceq)$.

Now we derive a relation $\leq$ over $\mathcal{L}$ from the pre-order $\preceq$ as follows:

For $\phi, \psi \in \mathcal{L}$, $\phi \leq \psi$ iff $min(|\neg\phi|, \preceq) \preceq min(|\neg\psi|, \preceq)$

It has been shown that the relation $\leq$ derived from $\preceq$ as above is an epistemic entrenchment [26]. Thus $\leq$ can be used to determine an EHC function via $(HC\dot{-})$. Let $-$ be the EHC function for H that is determined by $\leq$. It remains to show $H\dot{-}\phi = H - \phi$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$.

For one direction, suppose $\psi \in H\dot{-}\phi$. We need to show $\psi \in H-\phi$. By $(SHC\dot{-})$, $\psi \in H\dot{-}\phi$ implies $\psi \in H$ and either $\vdash \phi$ or there is $\chi \in \mathcal{HS}(\phi \vee \psi)$ such that $min(|\neg\phi|, \preceq) \prec min(|\neg\chi|, \preceq)$. By the derivation of $\leq$, the last part implies $\phi < \chi$. Thus it follows from $(HC\dot{-})$ that $\psi \in H - \phi$. The other direction can be proved in a similar manner. □

Proof for Theorem 6 Suppose $|H| \cap |\phi| \neq \emptyset$. By the construction of SHCIR, $|H * \phi| = Cl_{\cap}(|H\dot{-}\chi_1| \cup \cdots \cup |H\dot{-}\chi_n|) \cap |\phi|$ where $\dot{-}$ is the SHCMHC function that generates $*$ and $\mathcal{HS}(\neg\phi) = \{\chi_1, \ldots, \chi_n\}$. By the definition of Horn strengthenings we have $|\chi_i| \subseteq |\neg\phi|$ which implies $|\phi| \subseteq |\neg\chi_i|$ for $1 \leq i \leq n$. Thus $|H| \cap |\neg\chi_i| \neq \emptyset$ which implies $H \nvdash \chi_i$. It then follows from $(H\dot{-}3)$ that $H\dot{-}\chi_i = H$. Thus $|H * \phi| = |H| \cap |\phi|$. Then by the faithfulness of $\preceq$, $|H| \cap |\phi| = min(|\phi|, \preceq)$.

Now suppose $|H| \cap |\phi| = \emptyset$. For one direction, suppose $\mu \in |H * \phi|$. We need to show $\mu \in min(|\phi|, \preceq)$. By the definition of SHCIR, $|H * \phi| = Cl_{\cap}(Cl_{\cap}(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_{\cap}(H \cup min(|\neg\chi_n|, \preceq))) \cap |\phi|$. Since $\mu \in |H * \phi|$, by Lemma 5 there is $\nu \in Cl_{\cap}(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_{\cap}(H \cup min(|\neg\chi_n|, \preceq))$ such that $\mu \preceq \nu$. Thus there is χ_i such that $\nu \in Cl_{\cap}(H \cup min(|\neg\chi_i|, \preceq))$. Again, by Lemma 5, there is $\sigma \in H \cup min(|\neg\chi_i|, \preceq)$ such that $\nu \preceq \sigma$. By the faithfulness of $\preceq$ we cannot have $\sigma \in |H|$, thus it must be that $\sigma \in min(|\neg\chi_i|, \preceq)$. Since $|\phi| \subseteq |\neg\chi_i|$, $\sigma \in min(|\neg\chi_i|, \preceq)$ implies $\sigma \preceq x$ for all $x \in |\phi|$. By the transitivity of $\preceq$, it follows from $\mu \preceq \nu$ and $\nu \preceq \sigma$ that $\mu \preceq \sigma$. Thus we have $\mu \preceq x$ for all $x \in |\phi|$ which implies $\mu \in min(|\phi|, \preceq)$.

For the other direction, suppose $\mu \in min(|\phi|, \preceq)$. We need to show $\mu \in |H * \phi|$. Assume to the contrary that $\mu \notin |H * \phi|$. Recall that $|H * \phi| = Cl_{\cap}(Cl_{\cap}(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_{\cap}(H \cup min(|\neg\chi_n|, \preceq))) \cap |\phi|$. We first show that the assumption leads to $|H * \phi| \cap |\phi| = \emptyset$.

Assume there is $\omega \in |\phi|$ such that $\omega \in |H * \phi|$ which implies $\omega \in Cl_{\cap}(Cl_{\cap}(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_{\cap}(H \cup min(|\neg\chi_n|, \preceq)))$. Again, by Lemma 5, there is $\sigma \in min(|\neg\chi_i|, \preceq)$ such that $\omega \preceq \sigma$ for some $\chi_i \in \mathcal{HS}(\neg\phi)$. Since $\mu \in min(|\phi|, \preceq)$ and $\omega \in |\phi|$, we have $\mu \preceq \omega$. It then follows from $\omega \preceq \sigma$ and $\mu \preceq \omega$ that $\mu \preceq \omega$ which implies $\mu \in |\phi| \subseteq |\neg\chi_i|$, a contradiction. Thus we have $|H * \phi| \cap |\phi| = \emptyset$.

Now Let $\omega \in min(|\neg\chi_i|, \preceq)$ for some $\chi_i \in \mathcal{HS}(\neg\phi)$. Then there is $\chi_j \in \mathcal{HS}(\neg\phi)$ such that $\omega \notin |\neg\chi_j|$ for otherwise it follows from Lemma 2 that $\omega \in |\phi|$ which contradicts $|H * \phi| \cap |\phi| = \emptyset$. Let $\nu \in min(|\neg\chi_j|, \preceq)$. Since $\omega \in |\chi_j|$, $\nu \notin |\chi_j|$, and $\nu \in |\neg\phi|$, we have by the definition of Horn strengthening that $\omega \cap \nu \in |\phi|$. Since $\omega, \nu \in Cl_\cap(Cl_\cap(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_\cap(H \cup min(|\neg\chi_n|, \preceq)))$, we have $\omega \cap \nu \in Cl_\cap(Cl_\cap(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_\cap(H \cup min(|\neg\chi_n|, \preceq)))$ which implies $|H * \phi| \cap |\phi| \neq \emptyset$, a contradiction. $\square$

Proof for Theorem 8 Let $\phi \in \mathcal{L}_{\mathrm{H}}$ be such that $\mathcal{HS}(\neg\phi) = \{\chi_1, \ldots, \chi_n\}$. By Lemma 2 we have $|\neg\phi| = |\chi_1| \cup \cdots \cup |\chi_n|$ which implies, by De Morgan's laws, $|\phi| = |\neg\chi_1| \cap \cdots \cap |\neg\chi_n|$. Assume to the contrary that $min(|\neg\chi|, \preceq) \cap |\phi| = \emptyset$ for all $\chi \in \mathcal{HS}(\neg\phi)$. Let $\mu \in min(|\neg\chi_i|, \preceq)$ for some $\chi_i \in \mathcal{HS}(\neg\phi)$, then there is $\chi_j \in \mathcal{HS}(\neg\phi)$ such that $\mu \notin |\neg\chi_j|$ for otherwise it follows from $|\phi| = |\neg\chi_1| \cap \cdots \cap |\neg\chi_n|$ that $\mu \in |\phi|$ which contradicts the fact that $min(|\neg\chi|, \preceq) \cap |\phi| = \emptyset$ for all $\chi \in \mathcal{HS}(\neg\phi)$. Let $\nu \in min(|\neg\chi_j|, \preceq)$. Since $\mu \in |\chi_j|$, $\nu \notin |\chi_j|$ and $\nu \in |\neg\phi|$, we have by the definition of Horn strengthening that $\mu \cap \nu \in |\phi|$. It then follows from $|\phi| = |\neg\chi_1| \cap \cdots \cap |\neg\chi_n|$, that $\mu \cap \nu \in |\neg\chi_i|$ and $\mu \cap \nu \in |\neg\chi_j|$. Due to the strict Horn compliance of $\preceq$, we have either $\mu \cap \nu \preceq \mu$ or $\mu \cap \nu \preceq \nu$ which implies either $\mu \cap \nu \in min(|\neg\chi_i, \preceq)|$ or $\mu \cap \nu \in min(|\neg\chi_j, \preceq)|$, a contradiction. Thus there is $\chi \in \mathcal{HS}(\neg\phi)$ such that $min(|\neg\chi|, \preceq) \cap |\phi| \neq \emptyset$.

Let's assume, without loss of generality, that $\chi_1 \in \mathcal{HS}(\neg\phi)$ is such that $min(|\neg\chi_1|, \preceq) \cap |\phi| \neq \emptyset$. By Theorem 6 we have $|H * \phi| = min(|\phi|, \preceq)$. Thus it suffices to show $|H \dot{-} \chi_1 + \phi| = min(|\phi|, \preceq)$. For one direction, we have, by the construction of MHC and IR, that $|H \dot{-} \chi_1 + \phi| = Cl_\cap(|H| \cup min(|\neg\chi_1|, \preceq)) \cap |\phi| \subseteq Cl_\cap(Cl_\cap(H \cup min(|\neg\chi_1|, \preceq)) \cup \cdots \cup Cl_\cap(H \cup min(|\neg\chi_n|, \preceq))) \cap |\phi| = |H * \phi| = min(|\phi|, \preceq)$. For the other direction, suppose $\mu \in min(|\phi|, \preceq)$. Let $\nu \in min(|\neg\chi_1|, \preceq) \cap |\phi|$. Then we have $\mu \preceq \nu$. Since $|\phi| \subseteq |\neg\chi_1|$, we have $\mu \in |\neg\chi_1|$. Thus it follows from $\nu \in min(|\neg\chi_1|, \preceq)$ and $\mu \preceq \nu$ that $\mu \in min(|\neg\chi_1|, \preceq)$. Thus $min(|\phi|, \preceq) \subseteq min(|\neg\chi_1|, \preceq) \cap |\phi| \subseteq Cl_\cap(|H| \cup min_\preceq|\neg\chi_1|) \cap |\phi| = |H - \chi_1 + \phi|$. $\square$

Proof for Lemma 6 Suppose $\mu, \nu \in |\phi|$ and $\mu \preceq \nu$. Since $\mathcal{HS}(\neg\phi) = \{\chi_1, \chi_2, \ldots, \chi_n\}$, we have by the definition of Horn strengthenings that $|\chi_i| \subseteq |\neg\phi|$ which implies $|\phi| \subseteq |\neg\chi_i|$ for $1 \leq i \leq n$. We first show $\mu \preceq_{\chi_1}^{-1} \nu$. There are three cases:

Case 1, $\mu, \nu \notin Cl_\cap(|H| \cup min(|\neg\chi_1|, \preceq))$: $\mu \preceq_{\chi_1}^{-1} \nu$ follows immediately from HC2.

Case 2, $\mu \in Cl_\cap(|H| \cup min(|\neg\chi_1|, \preceq))$: It follows from HC1 that μ is minimal in $\preceq_{\chi_1}^{-1}$. Thus $\mu \preceq_{\chi_1}^{-1} \nu$.

Case 3, $\nu \in Cl_\cap(|H| \cup min(|\neg\chi_1|, \preceq))$ and $\mu \notin Cl_\cap(|H| \cup min(|\neg\chi_1|, \preceq))$: If $\nu \in |H| \cup min(|\neg\chi_1|, \preceq)$, then $\mu \preceq \nu$ implies $\mu \in |H| \cup min(|\neg\chi_1|, \preceq)$, a contradiction. If $\nu \notin |H| \cup min(|\neg\chi_1|, \preceq)$, then by Lemma 5, there is $\delta \in |H| \cup min(|\neg\chi_1|, \preceq)$ such that $\nu \preceq \delta$. It then follows from $\mu \preceq \nu$ and the transitivity of $\preceq$ that $\mu \preceq \delta$ which implies $\mu \in |H| \cup min(|\neg\chi_1|, \preceq)$, a contradiction.

$\mu \preceq^{-i}_{\chi_i} \nu$ for $2 \leq i \leq n$ can be proved inductively in the same manner as for $\preceq^{-1}_{\chi_1}$. The proof for the opposite direction is similar. □

Proof for Theorem 9 Suppose $\mathcal{HS}(\neg\phi) = \{\chi_1, \chi_2, \ldots, \chi_n\}$. Then by the definition of Horn strengthenings we have $|\chi_i| \subseteq |\neg\phi|$ which implies $|\phi| \subseteq |\neg\chi_i|$ for $1 \leq i \leq n$. It follows from the definition of SHCSR that $|H * \phi| = |((\cdots((H -_1 \chi_1) -_2 \chi_2)\cdots) -_n \chi_n) + \phi| = Cl_\cap(|H| \cup min(|\neg\chi_1|, \preceq) \cup min(|\neg\chi_2|, \preceq^{-1}_{\chi_1}) \cup \cdots \cup min(|\neg\chi_n|, \preceq^{-n-1}_{\chi_{n-1}})) \cap |\phi|$ where $-_1$ is the SHCMHC function that generates $*$. Note that HC1 and HC2 are used for deriving all the posterior pre-orders $\preceq^{-1}_{\chi_1}, \ldots, \preceq^{-n-1}_{\chi_{n-1}}$ which in turn determine the MHC functions $-_2, \ldots, -_n$. Let $\preceq^{-0}_{\chi_0} = \preceq$.

For one direction, suppose $\omega \in |H * \phi|$, we need to show $\omega \in min(|\phi|, \preceq)$ There are three cases:

Case 1, $\omega \in |H|$: It follows from the faithfulness of $\preceq$ that $\omega \in min(|\phi|, \preceq)$.

Case 2, there is χ_i such that $\omega \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$: Since $\omega \in |\phi| \subseteq |\neg\chi_i|$, $\omega \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$ implies $\omega \in min(|\phi|, \preceq^{-i-1}_{\chi_{i-1}})$. Since $min(|\phi|, \preceq) = min(|\phi|, \preceq^{-i-1}_{\chi_{i-1}})$ follows from Lemma 6, we have $\omega \in min(|\phi|, \preceq)$.

Case 3, ω is induced by models in $|H| \cup min(|\neg\chi_1|, \preceq) \cup min(|\neg\chi_2|, \preceq^{-1}_{\chi_1}) \cup \cdots \cup min(|\neg\chi_n|, \preceq^{-n-1}_{\chi_{n-1}})$: Since $\preceq$ satisfies SHC, it follows from Lemma 5 that there is $\mu \in |H| \cup min(|\neg\chi_1|, \preceq) \cup min(|\neg\chi_2|, \preceq^{-1}_{\chi_1}) \cup \cdots \cup min(|\neg\chi_n|, \preceq^{-n-1}_{\chi_{n-1}})$ such that $\omega \preceq \mu$. If $\mu \in |H|$, then it follows from $\omega \notin |H|$ and the faithfulness of $\preceq$ that $\mu \prec \omega$ which contradicts $\omega \preceq \mu$. So there is χ_i such that $\mu \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$. Due to HC1 and HC2, the rankings for models of ϕ are not downgraded throughout the contraction sequence. It then follows from $\omega \in |\phi|$ and $\omega \preceq \mu$ that $\omega \preceq^{-i-1}_{\chi_{i-1}} \mu$. Thus $\omega \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$ which implies, as in Case 2, $\omega \in min(|\phi|, \preceq)$.

For the other direction, suppose $\omega \in min(|\phi|, \preceq)$, we need to show $\omega \in |H * \phi|$. Assume $\omega \notin |H * \phi|$. We first show that the assumption implies $|H * \phi| \cap |\phi| = \emptyset$. Assume there is $\mu \in |\phi|$ such that $\mu \in |H * \phi|$. There are three cases.

Case 1, $\mu \in |H|$: It follows from the faithfulness of $\preceq$ and $\omega \preceq \mu$ that $\omega \in |H|$, a contradiction.

Case 2, there is χ_i such that $\mu \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$: It follows from Lemma 6 and $\omega \preceq \mu$ that $\omega \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$, a contradiction.

Case 3, μ is induced by models in $|H| \cup min(|\neg\chi_1|, \preceq) \cup min(|\neg\chi_2|, \preceq^{-1}_{\chi_1}) \cup \cdots \cup min(|\neg\chi_n|, \preceq^{-n-1}_{\chi_{n-1}})$: Since $\preceq$ satisfies SHC, it follows from Lemma 5 that there is $\nu \in |H| \cup min(|\neg\chi_1|, \preceq) \cup min(|\neg\chi_2|, \preceq^{-1}_{\chi_1}) \cup \cdots \cup min(|\neg\chi_n|, \preceq^{-n-1}_{\chi_{n-1}})$ such that $\mu \preceq \nu$. It then follows from $\omega \preceq \mu$ and the transitivity of $\preceq$ that $\omega \preceq \nu$. Then, with the same reasoning as in the Case 3 above, we can derive a contradiction.

Since all cases lead to a contradiction, we have $|H * \phi| \cap |\phi| = \emptyset$. Let $\mu \in min(|\neg\chi_i|, \preceq^{-i-1}_{\chi_{i-1}})$. Then there is χ_j such that $\mu \notin |\neg\chi_j|$ for otherwise $\mu \in |\phi|$

which leads to $|H * \phi| \cap |\phi| \neq \emptyset$. Let $\nu \in min(|\neg\chi_j|, \preceq^{-j-1}_{\chi_{j-1}})$. Since $\mu \in |\chi_j|$, $\nu \notin |\chi_j|$, and $\nu \in |\neg\phi|$, we have by the definition of Horn strengthening that $\mu \cap \nu \in |\phi|$. Since $\mu, \nu \in |H * \phi| = Cl_\cap(|H * \phi|)$, we have $\mu \cap \nu \in |H * \phi|$ which implies $|H * \phi| \cap |\phi| \neq \emptyset$, a contradiction. □

Proof for Lemma 7 For one direction suppose $\omega \in min(|\phi|, \preceq)$, we need to show $\omega \in min(|\chi_1|, \preceq) \cup \cdots \cup min(|\chi_n|, \preceq)$. By Lemma 2 there is $\chi_i \in \mathcal{HS}(\phi)$ such that $\omega \in |\chi_i|$. By the definition of Horn strengthening, $|\chi| \subseteq |\phi|$ for all Horn strengthenings χ of ϕ. Thus there is no χ such that $\mu \in |\chi|$ and $\mu \prec \omega$ which implies $\chi_i \in min(\mathcal{HS}(\phi), \preceq)$ and we are done.

For the other direction, suppose $\omega \in min(|\chi_1|, \preceq) \cup \cdots \cup min(|\chi_n|, \preceq)$, we need to show $\omega \in min(|\phi|, \preceq)$. Without loss of generality, let $\omega \in min(|\chi_1|, \preceq)$. Since $|\chi_1| \subseteq |\phi|$, we have $\omega \in |\phi|$. By the first part of the proof, we have $min(|\chi|, \preceq) \subseteq min(|\chi_1|, \preceq) \cup \cdots \cup min(|\chi_n|, \preceq)$. Thus it follows from the definition of most preferred formulas and $\omega \in min(|\chi_1|, \preceq)$ that $\mu \not\prec \omega$ for all $\mu \in min(|\chi|, \preceq)$ which implies $\omega \in min(|\phi|, \preceq)$. □

Proof for Theorem 11 For one direction, suppose $\dot{-}$ is a MHC function for H that is determined by the pre-order $\preceq$, we need to show $\dot{-}$ is an IC function for H. Suppose $*$ is the MHR function for H that is determined by $\preceq$. Let $-$ be the IC function generated by $*$. It suffices to show $|H \dot{-} \phi| = |H - \phi|$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$. Let $min(\mathcal{HS}(\neg\phi), \preceq) = \{\chi_1, \ldots, \chi_n\}$. By the definition of MHC and IC functions, we have $|H \dot{-} \phi| = Cl_\cap(|H| \cup min(|\neg\phi|, \preceq))$ and $|H - \phi| = Cl_\cap(|H| \cup min(|\chi_1|, \preceq), \ldots, min(|\chi_n|, \preceq))$. Thus it suffices to show $|H| \cup min(|\neg\phi|, \preceq) = |H| \cup min(|\chi_1|, \preceq), \ldots, min(|\chi_n|, \preceq)$ which follows immediately from Lemma 7.

For the other direction, suppose $\dot{-}$ is an IC function for H that is generated from the MHR function $*$ for H, we need to show $\dot{-}$ is a MHC function for H. Suppose the determining pre-order for $*$ is $\preceq$. Now suppose $-$ is a MHC function for H that is determined by $\preceq$. It suffices to show $|H \dot{-} \phi| = |H - \phi|$ for all $\phi \in \mathcal{L}_{\mathrm{H}}$. As in the first part of the proof, this comes down to showing $|H| \cup min(|\neg\phi|, \preceq) = |H| \cup min(|\chi_1|, \preceq), \ldots, min(|\chi_n|, \preceq)$ which follows immediately from Lemma 7. □

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