Expressiveness of Logic Programs under the General Stable Model Semantics

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The stable model semantics had been recently generalized to non-Herbrand structures by several works, which provides a unified framework and solid logical foundations for answer set programming. This paper focuses on the expressiveness of normal and disjunctive logic programs under the general stable model semantics. A translation from disjunctive logic programs to normal logic programs is proposed for infinite structures. Over finite structures, some disjunctive logic programs are proved to be intranslatable to normal logic programs if the arities of auxiliary predicates and functions are bounded in a certain way. The equivalence of the expressiveness of normal logic programs and disjunctive logic programs over arbitrary structures is also shown to coincide with that over finite structures, and coincide with whether NP is closed under complement. Moreover, to obtain a more explicit picture of the expressiveness, some intertranslatability results between logic program classes and fragments of second-order logic are established.

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1. INTRODUCTION

Logic programming with default negation is an elegant and efficient formalism for Knowledge Representation and Reasoning, which incorporates the abilities of classical logic, inductive definition and commonsense reasoning. Nowadays, the most prominent semantics for this formalism is the stable model semantics proposed by Gelfond and Lifschitz [Gelfond and Lifschitz 1988]. Logic programming based on this semantics, which is known as Answer Set Programming (ASP), has then emerged as a flourishing paradigm for declarative programming in the last two decades.

The original stable model semantics focuses only on Herbrand structures in which the unique name assumption is made. For a certain class of applications, this assumption will simplify the representation. However, there are many applications where the knowledge can be more naturally represented over non-Herbrand structures including arithmetical
structures. To overcome this limit, the general stable model semantics, which generalizes the original semantics to arbitrary structures, was then proposed via second-order logic [Ferraris et al. 2011], via circumscription [Lin and Zhou 2011], and via Gödel’s 3-valued logic [Pearce and Valverde 2005], respectively, which provides us a unified framework for answer set programming, armed with powerful tools from classical logic.

The main goal of this work is to identify the expressiveness of logic programs, which is one of the central topics in Knowledge Representation and Reasoning. We will focus on two important classes of logic programs – normal logic programs (NLP) and disjunctive logic programs (DLP). Over Herbrand structures, the expressiveness of logic programs under query equivalence has been thoroughly studied in the last three decades. A comprehensive survey for these works can be found in [Dantsin et al. 2001]. Our task is quite different here. On the one hand, we will work on the general stable model semantics so that non-Herbrand structures will be considered. On the other hand, instead of considering query equivalence, the expressiveness in our work will be based on model equivalence. This setting is important since ASP solvers are usually used to generate models. Note that model equivalence always implies query equivalence, but the converse is in general false.

We also hope this work contributes to the effective implementation of answer set solvers. Translating logic programs into classical logics is a usual approach to implement answer set solvers. For example, in the propositional case, there have been a number of works that implemented answer set solving by reducing the existence of answer sets to the satisfiability of classical propositional logic, see, e.g., [Lin and Zhao 2004; Lierler and Maratea 2004]. In this work, we are interested in translating normal logic programs to first-order sentences so that the state-of-the-art SMT solvers can be used for answer set solving. On the other hand, our work also considers the optimization of logic programs. It is clear that a language is simpler in the syntax, then it is more likely to have an efficient implementation. Therefore, in this work, we will also investigate whether a rich language can be compiled (translated) to a simple language or not. In particular, since the arity of auxiliary symbol is the most important factor to introduce nondeterminism [Immerman 1999], we will try to find translations in which the maximum arity of auxiliary symbols is as small as possible.

Our contribution in this paper is fourfold. Firstly, we show that, over infinite structures, every disjunctive program can be equivalently translated to a normal one. Secondly, we prove that, if only finite structures are considered, for each integer \( n \) greater than 1 there is a disjunctive program with intensional predicates of arities less than \( n \) that cannot be equivalently translated to any normal program with auxiliary predicates of arities less than \( 2n \). Thirdly, we show that disjunctive and normal programs are of the same expressiveness over arbitrary structures if, and only if, they are of the same expressiveness over finite structures, if, and only if, the complexity class NP is closed under complement. Lastly, to understand the exact expressiveness of logic programs, we also prove that the intertranslatability holds between some classes of logic programs and some fragments of second-order logic.

The rest of this paper is organized as follows. Section 2 presents necessary concepts, notions, definitions and background knowledge that we will need through out this paper. Section 3 studies the expressiveness of logic programs over infinite structures where we propose a translation from DLP to NLP. Section 4 then focuses the expressiveness over finite structures. In particular, we show that over finite structures, DLP and NLP have the same expressiveness if, and only if, \( \text{NP} \) is closed under complement. A more subtle intranslatability property from DLP to NLP is also proved in this section. Based on the
results from Sections 3 and 4, Section 5 compares the expressiveness of NLP and DLP over arbitrary structures. Finally, Section 6 concludes the paper with some remarks.

2. PRELIMINARIES

A vocabulary consists of a finite set of predicate constants and a finite set of function constants. Logical symbols are as usual, including a countable set of predicate variables and a countable set of function variables. Every constant or variable is equipped with a natural number, called its arity. Nullary function constants and variables are called individual constants and variables, respectively. Nullary predicate constants are called propositional constants. Sometimes we do not distinguish between predicate constants and predicate variables, and simply call them predicates; and likewise we sometimes refer to function constants and function variables as functions if no confusion occurs. Atoms, formulas, sentences and theories of a vocabulary $\mathcal{V}$ (or shortly, $\mathcal{V}$-atoms, $\mathcal{V}$-formulas, $\mathcal{V}$-sentences and $\mathcal{V}$-theories) are built from constants in $\mathcal{V}$, variables, equality $=$, connectives $\bot, \top, \land, \lor, \rightarrow$, and quantifiers $\exists, \forall$ in a standard way. Every positive clause of $\mathcal{V}$-atoms. Given a sentence $\psi$ and a theory $\Sigma$, let $\psi(\phi)$ and $\psi(\Sigma)$ denote the sets of predicate and function constants that occur in $\phi$ and $\Sigma$, respectively.

Suppose $Q \in \{\forall, \exists\}$, $\tau = \{X_1, \ldots, X_n\}$, and $\vec{x} = x_1 \cdots x_m$, where $X_i$ ranges over predicate and function variables, and $x_j$ ranges over individual variables. We let $Q\tau$ and $Q\vec{x}$ be shorthands of the quantifier blocks $Q\tau_1 \cdots Q\tau_n \phi$, where $Q_i$ is $\exists$ if $i$ is odd, and $\forall$ otherwise; $\tau_i$ is a finite set of variables of arities $\leq k$; and no second-order quantifier appears in $\phi$. Let $\Sigma_{n,k}$ denote the class defined the same as $\Sigma_{n,k}^1$ except no function variable allowed in any $\tau_i$. Let $\Sigma_{n,k}^1$ (respectively, $\Sigma_{n,k}^0$) be the union of $\Sigma_{n,k}^{1,i}$ (respectively, $\Sigma_{n,k}^0$) for all $k \geq 0$. Given a class $\Lambda$ defined as above, let $\Lambda[\forall^*\exists^*]$ (respectively, $\Lambda[\forall^*]$) be the class of sentences in $\Lambda$ with first-order part of the form $\forall \vec{x} \exists \vec{y} \theta$ (respectively, $\forall \vec{x} \theta$), where $\vec{x}, \vec{y}$ are tuples of individual variables, and $\theta$ is quantifier-free.

Example 1. Let $\phi$ denote the second-order formula

$$\exists X \forall y (X(a) \land (X(x) \land E(x, y) \rightarrow X(y)) \land \neg X(b))$$

where $E$ is a binary predicate constant, $X$ is a unary predicate variable, and $b, c$ are two individual constants. As defined, it is clear that $\phi$ is a $\Sigma_{1}^{1}$-sentence and, moreover, it is also in $\Sigma_{1}^{1}$. Towards a more explicit classification, $\phi$ is also a $\Sigma_{1,1}^{1} [\forall^*\exists^*]$-sentence because the first-order part is universal, and only one unary predicate variable is quantified.

Example 2. Let $\psi$ denote the second-order formula

$$\exists f (\forall \vec{xy} (f(x) = f(y) \rightarrow x = y) \land \exists x \forall y -(x = f(y)))$$

where $f$ is a unary function variable. Clearly, $\psi$ is a $\Sigma_{1}^{1}$-sentence, but it is not in $\Sigma_{1}^{1}$.

Every structure $A$ of $\mathcal{V}$ (or shortly, $\mathcal{V}$-structure $A$) is accompanied by a nonempty set $A$, called the domain of $A$, and interprets each $n$-ary predicate constant $P$ in $\mathcal{V}$ as an $n$-ary relation, denoted $PA$, on $A$, and interprets each $n$-ary function constant $f$ in $\mathcal{V}$ as an $n$-ary function, denoted $f^A$, on $A$. A structure is finite if its domain is finite, and infinite otherwise. Let FIN denote the class of finite structures, and INF denote the class of infinite
structures. A restriction of a structure $A$ to a vocabulary $\sigma$ is the structure obtained from $A$ by discarding all interpretations for constants which are not in $\sigma$. Given a vocabulary $\nu \supset \sigma$ and a $\sigma$-structure $B$, every $\nu$-expansion of $B$ is a structure $A$ of $\nu$ such that $B$ is a restriction of $A$ to $\sigma$. Given a structure $A$ and a set $\tau$ of predicates, let $\text{INS}(A, \tau)$ denote the set of ground atoms $P(\vec{a})$ for all tuples $\vec{a} \in P^A$ and all predicate constants $P$ in $\tau$.

Every assignment in a structure $A$ is a function that maps each individual variable to an element of $A$ and that maps each predicate (respectively, function) variable to a relation (respectively, function) on $A$ of the same arity. For convenience, we assume that the definition of assignments extends to terms naturally. Given a formula $\varphi$ and an assignment $\alpha$ in $A$, we write $A \models \varphi[\alpha]$ if $\alpha$ satisfies $\varphi$ in $A$ in the standard way. In particular, if $\varphi$ is a sentence, we simply write $A \models \varphi$ and say that $A$ is a model of $\varphi$, or in other words, $\varphi$ is true in $A$. We use $\text{Mod}(\varphi)$ to denote the set of all models of $\varphi$. Given formulas $\varphi, \psi$ and a class $C$ of structures, we say $\varphi$ is equivalent to $\psi$ over $C$, or write $\varphi \equiv_C \psi$ for short, if for every $A$ in $C$ and every assignment $\alpha$ in $A$, $\alpha$ satisfies $\varphi$ in $A$ if, and only if, $\alpha$ satisfies $\psi$ in $A$. In particular, if $C$ is the class of arbitrary structures, the words “over $C$” and the subscript $C$ can be dropped. Given a quantifier-free formula $\varphi$ and an assignment $\alpha$ in $A$, let $\varphi[\alpha]$ denote the ground formula obtained from $\varphi$ by (i) substituting $P(\alpha(t_1), \ldots, \alpha(t_k))$ for $P(t_1, \ldots, t_k)$ if $P(t_1, \ldots, t_k)$ is an atomic formula and $P$ is a predicate constant of arity $k$, followed by (ii) substituting $\top$ for $t_1 = t_2$ if $\alpha(t_1) = \alpha(t_2)$, and by (iii) substituting $\bot$ for $t_1 = t_2$ otherwise.

A class of structures is also called a property. Let $C$ and $D$ be any two properties. We say $D$ is defined by a sentence $\varphi$ over $C$, or equivalently, $\varphi$ defines $D$ over $C$, if each structure $A$ from $C$ is in $D$ if, and only if, $A$ is a model of $\varphi$; that is, $D = \text{Mod}(\varphi) \cap C$. $D$ is definable in a class $\Sigma$ of sentences over $C$ if there is a sentence in $\Sigma$ that defines $D$ over $C$. Given two classes $\Sigma$, $\Lambda$ of sentences, we write $\Sigma \leq_C \Lambda$ if each property definable in $\Sigma$ over $C$ is also definable in $\Lambda$ over $C$; we write $\Sigma \preceq_C \Lambda$ if both $\Sigma \leq_C \Lambda$ and $\Lambda \leq_C \Sigma$ hold. Intuitively, $\Sigma \leq_C \Lambda$ asserts that, over the structure class $C$, the fragment $\Lambda$ is at least as expressive as $\Sigma$; and $\Sigma \preceq_C \Lambda$ asserts that, over the structure class $C$, $\Sigma$ and $\Lambda$ are of the same expressiveness. Again, in above definitions, if $C$ is the class of arbitrary structures, the words “over $C$” and the subscript $C$ might be omitted.

**Example 3 (Example 1 continued).** Let $\varphi$ be the same as in Example 1, that is,

$$\varphi = \exists x \forall y (X(a) \land (X(x) \land E(x, y) \rightarrow X(y)) \land \neg X(b)). \quad (3)$$

Let $\nu$ denote the vocabulary $\{E, a, b\}$. It is clear that every finite $\nu$-structure $A$ can be regarded as a directed graph $G_A$ with two distinguished nodes $a^A$ and $b^A$ such that the node set of $G_A$ is the domain $A$ and the edge set of $G_A$ is the binary relation $E^A$.

Let Unreach denote the class of finite $\nu$-structures $A$ such that $b^A$ is unreachable from $a^A$ in the directed graph $G_A$. It is easy to see that every finite $\nu$-structure is in Unreach if, and only if, it is a model of $\varphi$; that is, $\varphi$ defines Unreach over the class of finite structures.

**Example 4 (Example 2 continued).** Let $\psi$ be the same as in Example 2, that is,

$$\psi = \exists f(\forall x y (f(x) = f(y) \rightarrow x = y) \land \exists x \forall y \neg(x = f(y))). \quad (4)$$

Let $\sigma$ denote the empty vocabulary $\emptyset$. One can check that every $\sigma$-structure $A$ is a model of $\psi$ if, and only if, $A$ is infinite. Thus, $\psi$ defines $\text{INF}$ over the class of arbitrary structures.
2.1 Logic Programs and Stable Models

Every disjunctive logic program (or simply called disjunctive program) consists of a set of rules, each of which is a first-order formula of the following form:

\[ \theta_1 \land \cdots \land \theta_m \rightarrow \theta_{m+1} \lor \cdots \lor \theta_n \tag{5} \]

where \( 0 \leq m \leq n \) and \( n > 0 \); \( \theta_i \) is a literal (i.e., an atom or the negation of an atom) if \( 1 \leq i \leq m \); \( \theta_i \) is an atom without involving any equality if \( m < i \leq n \); \( \theta_i \) is a literal (i.e., an atom or the negation of an atom) if \( 1 \leq i \leq m \). Given a rule, the disjunctive part is called its head, and the conjunctive part is called its body. Given a disjunctive program \( \Pi \), a predicate is called intensional (w.r.t. \( \Pi \)) if it appears in the head of some rule in \( \Pi \), and extensional otherwise. A formula is called intensional (w.r.t. \( \Pi \)) if it does not involve any extensional predicate of \( \Pi \). Let \( \nu(\Pi) \) denote the set of predicate and function constants that appear in \( \Pi \). Note that, in this paper, all negations in intensional literals w.r.t. \( \Pi \) will be assumed as default negations.

Let \( \Pi \) be a disjunctive program. Then \( \Pi \) is called normal if the head of each rule contains at most one atom, \( \Pi \) is plain if the negation of any intensional atom does not appear in the body of any rule, \( \Pi \) is propositional if it does not involve any predicate of a positive arity, and \( \Pi \) is finite if it contains only a finite set of rules. In general, we will assume that all disjunctive program in this paper are finite when they involve predicates with positive arities, while propositional logic programs may contain infinite sets of rules.

**Example 5.** Let \( \Pi \) be a disjunctive program consisting of the following rules

\[ P(a), \tag{6} \]
\[ P(x) \land E(x, y) \rightarrow P(y), \tag{7} \]
\[ \neg P(b) \rightarrow P(b), \tag{8} \]

where \( P \) and \( E \) are unary and binary predicates, respectively, and both \( a, b \) are individual constants. Clearly, \( \Pi \) is normal, and \( P \) is the only intensional predicate of \( \Pi \). Note that the body of the rule (6) is empty. In this case, we usually omit the connective \( \rightarrow \).

Given any disjunctive program \( \Pi \), let \( \text{SM}(\Pi) \) denote the second-order sentence

\[ \hat{\Pi} \land \forall \tau^*(\tau^* < \tau \rightarrow \neg \hat{\Pi}^*) \tag{9} \]

where:

1. \( \tau \) is the set of all intensional predicates w.r.t. \( \Pi \);
2. \( \tau^* \) is the set of predicate variables \( P^* \) for all predicates \( P \in \tau \), where for each predicate constant \( Q \in \tau \) we introduce a fresh predicate variable \( Q^* \);
3. \( \tau^* < \tau \) denotes the formula
\[ \bigwedge_{P \in \tau} \forall \vec{x}(P^*(\vec{x}) \rightarrow P(\vec{x})) \land \neg \bigwedge_{P \in \tau} \forall \vec{x}(P(\vec{x}) \rightarrow P^*(\vec{x})); \tag{10} \]
4. \( \hat{\Pi} \) is the conjunction of sentences \( \gamma^\nu \) for all rules \( \gamma \in \Pi \) where \( \psi^\nu \) denotes the first-order universal closure of \( \psi \) whenever \( \psi \) is a rule;

\(^1\)Note that, in this paper, we do not require the rules to be safe. The main reason is as follows: The general stable model semantics does not rely on the grounding technique, and an intended domain will be always specified; thus, the termination of the computation on a logic program does not depend on the safety of this program.
Let \( \overline{\Pi}^+ \) denote the conjunction of \( \gamma^\tau \) for all \( \gamma \in \Pi \) where each \( \gamma^\tau \) is the formula obtained from \( \gamma \) by substituting \( P^* \) for all positive occurrences of \( P \) in the head or in the body.

A structure \( A \) is called a stable model of \( \Pi \) if it is a model of \( \text{SM}(\Pi) \). For more details about the transformational semantics, please refer to [Ferraris et al. 2011]. Note that, since we only consider disjunctive programs, the transformation presented here is slightly simpler than the transformation in [Ferraris et al. 2011] where first-order sentences are focused. For the class of disjunctive programs, the equivalence of both transformations can be easily proved.

Given two properties \( C \) and \( D \), we say \( D \) is defined by a disjunctive program \( \Pi \) over \( C \) via the set \( \tau \) of auxiliary constants if the formula \( \exists \tau \text{SM}(\Pi) \) defines \( D \) over \( C \), where \( \tau \) is a set of predicates and functions occurring in \( \Pi \). Given \( n \geq 0 \), let \( \text{DLP}_n \) (respectively, \( \text{DLP}^F_n \)) be the class of sentences \( \exists \tau \text{SM}(\Pi) \) for all disjunctive programs \( \Pi \) and all finite sets \( \tau \) of predicate (respectively, predicate and function) constants of arities \( \leq n \). Let \( \text{DLP} \) (respectively, \( \text{DLP}^F \)) be the union of \( \text{DLP}_n \) (respectively, \( \text{DLP}^F_n \)) for all integers \( n \geq 0 \). Furthermore, in above definitions, if \( \Pi \) is restricted to be normal, we then obtain the notations \( \text{NLP}_n \), \( \text{NLP}^F_n \), \( \text{NLP} \), and \( \text{NLP}^F \), respectively.

Example 6 (Example 5 continued). Let \( \Pi \) be the logic program that is presented in Example 5. Then \( \overline{\Pi} \) denotes the first-order formula

\[
P(a) \land \forall x y (P(x) \land E(x, y) \rightarrow P(y)) \land (\neg P(b) \rightarrow P(b)),
\]

and \( \overline{\Pi}^+ \) denotes the formula

\[
P^*(a) \land \forall x y (P^*(x) \land E(x, y) \rightarrow P^*(y)) \land (\neg P(b) \rightarrow P^*(b)).
\]

By definition, we know that \( \text{SM}(\Pi) \) is the second-order sentence

\[
\overline{\Pi} \land \forall P^* (\forall x (P^*(x) \rightarrow P(x)) \land \neg \forall x (P(x) \rightarrow P^*(x)) \rightarrow \neg \overline{\Pi}^+).
\]

Let \( \tau \) be the set \( \{P\} \), and let \( \nu \) denote the vocabulary \( \{E, a, b\} \). Then it is easy to see that every finite \( \nu \)-structure \( A \) is a model of \( \exists \nu \text{SM}(\Pi) \) if, and only if, \( A \in \text{Unreach} \) does not hold (which is defined in Example 3). Let Reach denote the complement of Unreach. Therefore, the class Reach is defined by \( \Pi \) via \( \tau \) (as the set of auxiliary constants).

Given a rule \( \gamma \), let \( \gamma^\nu \) be the set of conjuncts in the body of \( \gamma \) in which no intensional predicate positively occurs, and let \( \gamma^+ \) be the rule obtained from \( \gamma \) by removing all literals in \( \gamma^\nu \). Given a disjunctive program \( \Pi \) and a structure \( A \), let \( \Pi^A \) be the set of rules \( \gamma^+[\alpha] \) for all assignments \( \alpha \) in \( A \) and all rules \( \gamma \) in \( \Pi \) such that \( \alpha \) satisfies \( \gamma^\nu \) in \( A \). Now, \( \Pi^A \) can be regarded as a propositional program where each ground atom as a proposition. This procedure is called the first-order Gelfond-Lifschitz reduction due to the following result.

Proposition 1 (Proposition 4 in [Zhang and Zhang 2013b]). Let \( \Pi \) be a disjunctive program and \( \tau \) be the set of intensional predicates. Then an \( \nu(\Pi) \)-structure \( A \) is a stable model of \( \Pi \) iff INS(\( A \), \( \tau \)) is a minimal (w.r.t. the set inclusion) model of \( \Pi^A \).

To make the logic program more readable, now let us present a splitting lemma. Note that it directly follows from the first splitting lemma presented in [Ferraris et al. 2009].

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To simplify the presentation, we slightly abuse the notion without confusion: Each predicate (function) constant that appears in \( \tau \) in the formula \( \exists \tau \text{SM}(\Pi) \) is now regarded as a predicate (function) variable of the same arity.
Proposition 2 (Splitting Lemma in [Ferraris et al. 2009]). Let $\Pi$ be a disjunctive program, and let $\{\Pi_1, \Pi_2\}$ be a partition of $\Pi$ such that all intensional predicates of $\Pi_1$ is extensional w.r.t. $\Pi_2$ and that no extensional predicate of $\Pi_1$ has any occurrence in $\Pi_2$. Then we have that $\text{SM}(\Pi)$ is equivalent to $\text{SM}(\Pi_1) \land \text{SM}(\Pi_2)$.

2.2 Progression Semantics

In this subsection, we review a progression semantics proposed by Zhang and Zhang [2013b], which generalizes Lobo et al.’s fixed point semantics [1992] by allowing default negations and arbitrary structures.

Now, let us present the semantics. For convenience, two positive clauses that contain the same set of atoms will be regarded as the same. Let $\Pi$ be a propositional, possibly infinite and plain disjunctive program. Let $\text{PC}(\nu(\Pi))$ denote the set of all positive clauses of $\nu(\Pi)$ and take $\Lambda \subseteq \text{PC}(\nu(\Pi))$. We define $\Gamma_\Pi(\Lambda)$ as the following positive clause set:

$$
\begin{cases}
H \lor C_1 \lor \cdots \lor C_k \\
| \begin{array}{l}
k \geq 0 & & H, C_1, \ldots, C_k \in \text{PC}(\nu(\Pi)) \\
& \land \exists p_1, \ldots, p_k \in \nu(\Pi) & \text{s.t.} \\
& p_1 \land \cdots \land p_k \rightarrow H \in \Pi & \\
& C_1 \lor p_1, \ldots, C_k \lor p_k \in \Lambda
\end{array}
\end{cases}
$$

It is easy to check that $\Gamma_\Pi$ is a monotone operator on $\text{PC}(\nu(\Pi))$.

With this notation, a progression operator for first-order programs can be defined via the first-order Gelfond-Lifschitz reduction. Given any disjunctive program $\Pi$ and any $\nu(\Pi)$-structure $A$, we define $\Gamma_\Pi^A$ as the operator $\Gamma_\Pi^{\nu(A)}$; that is, let $\Gamma_\Pi^A(\Lambda)$ denote $\Gamma_\Pi^{\nu(A)}(\Lambda)$ for all positive clause set $\Lambda \subseteq \text{PC}(\nu(\Pi^A))$. Furthermore, we define

$$
\Gamma_\Pi^A \uparrow_n = \begin{cases}
\emptyset & \text{if } n = 0; \\
\Gamma_\Pi^A(\Gamma_\Pi^A \uparrow_{n-1}) & \text{if } n > 0.
\end{cases}
$$

Finally, let $\Gamma_\Pi^A \uparrow_\omega$ denote the union of $\Gamma_\Pi^A \uparrow_n$ for all integers $n \geq 0$. To illustrate these definitions, a simple example is given as follows.

Example 7. Let $\Pi$ be the disjunctive program consisting of the following rules:

$$
\begin{align*}
S(x) \lor T(x), \\
T(x) \land E(x, y) & \rightarrow T(y).
\end{align*}
$$

Let $\nu = \{E\}$, and let $A$ be a structure of $\nu$ defined as follows:

1. the domain $A$ is the set $\{a_i \mid i \in \mathbb{N}\}$ where $\mathbb{N}$ denotes the set of natural numbers;
2. the predicate $E$ is interpreted as the successor relation, i.e., $\{(a_i, a_{i+1}) \mid i \in \mathbb{N}\}$.

Then, $\Pi^A$ is a propositional disjunctive program consisting of the following rules:

$$
\begin{align*}
S(a_i) \lor T(a_i) & \quad (i \in \mathbb{N}), \\
T(a_i) & \rightarrow T(a_{i+1}) \quad (i \in \mathbb{N}).
\end{align*}
$$

Therefore, the progression of $\Pi$ on $A$ is then defined as follows:

$$
\begin{align*}
\Gamma_\Pi^A \uparrow_0 &= \{\}
\Gamma_\Pi^A \uparrow_1 &= \{S(a_i) \lor T(a_i) \mid i \in \mathbb{N}\},
\end{align*}
$$
\[ \vdots \]
\[ \Gamma^\mathbf{A}_{\Pi} \uparrow_n = \{ S(a_i) \lor T(a_j) \mid i, j \in \mathbb{N} \land i \leq j < i + n \}, \]

\[ \vdots \]
\[ \Gamma^\mathbf{A}_{\Pi} \uparrow_\omega = \{ S(a_i) \lor T(a_j) \mid i, j \in \mathbb{N} \land i \leq j \}. \]

The following proposition shows that the general stable model semantics can be equivalently redefined by the progression operator that we have defined.

**Theorem 1 (Theorem 1 in [Zhang and Zhang 2013b]).** Let \( \Pi \) be a disjunctive program, \( \tau \) be the set of intensional predicates of \( \Pi \), and \( \mathbf{A} \) be a structure of \( \upsilon(\Pi) \). Then \( \mathbf{A} \) is a stable model of \( \Pi \) iff \( \text{INS}(\mathbf{A}, \tau) \) is a minimal model of \( \Gamma^\mathbf{A}_{\Pi} \uparrow_\omega \).

**Remark 1.** In the above theorem, if \( \Pi \) is a normal program, we have that \( \mathbf{A} \) is a stable model of \( \Pi \) if, and only if, \( \text{INS}(\mathbf{A}, \tau) = \Gamma^\mathbf{A}_{\Pi} \uparrow_\omega \).

### 3. INFINITE STRUCTURES

Now we study the expressiveness of logic programs over infinite structures. We first propose a translation that reduces each disjunctive program to a normal program over infinite structures. The main idea is to encode ground positive clauses by elements in the intended domain. With the encoding, we then simulate the progression of the given disjunctive program by the progression of a normal program.

We first show how to encode a positive clause by an element. Let \( \mathbf{A} \) be an infinite set. Every *encoding function* on \( \mathbf{A} \) is an injective function from \( A \times A \) into \( A \). Let \( \text{enc} \) be an encoding function on \( \mathbf{A} \) and \( c \) an element in \( \mathbf{A} \) such that \( \text{enc}(a, b) \neq c \) for all elements \( a, b \in A \). To simplify the statement, let \( \text{enc}(a_1, \ldots, a_k; c) \) be short for the expression

\[ \text{enc}(\cdots \text{enc}(\text{enc}(c, a_1), a_2), \cdots, a_k) \]  

(20)

for any \( k \geq 0 \) and any set of elements \( a_1, \ldots, a_k \in A \). In the above expression, the special element \( c \) is used as a flag to indicate that the encoded tuple will be started after \( c \), and is then called the *encoding flag* of this encoding. Intuitively, by \( b = \text{enc}(a; c) \) we means that \( b \) is the encoding of \( a \) related to the encoding function \( \text{enc} \) and the encoding flag \( c \).

Let \( A^* \) denote the set of finite tuples of elements in \( A \), and let

\[ \text{enc}(A; c) = \{ b \in A \mid \exists \bar{a} \in A^* \text{ s.t. } b = \text{enc}(\bar{a}; c) \}. \]  

(21)

The *merging function* \( \text{mrg} \) on \( A \) related to \( \text{enc} \) and \( c \) is then defined as the function from \( \text{enc}(A; c) \times \text{enc}(A; c) \) into \( \text{enc}(A; c) \) such that

\[ \text{mrg}(\text{enc}(\bar{a}_1; c), \text{enc}(\bar{a}_2; c)) = \text{enc}(\bar{a}_1 \bar{a}_2; c) \]  

(22)

for all tuples \( \bar{a}_1 \) and \( \bar{a}_2 \) in \( A^* \). Suppose \( b_i = \text{enc}(\bar{a}_i; c) \) for \( 1 \leq i \leq k \). Again, to simplify the statement, we let \( \text{mrg}(b_1, \ldots, b_k) \) be short for the expression

\[ \text{mrg}(\cdots \text{mrg}(\text{mrg}(b_1, b_2), b_3), \cdots, b_k). \]  

(23)

It is easy to see that the following holds:

\[ \text{mrg}(b_1, \ldots, b_k) = \text{enc}(\bar{a}_1 \cdots \bar{a}_k; c). \]  

(24)
In other words, if \( b = \text{mrg}(b_1, \ldots, b_k) \), \( b \) is then an element that encodes the tuple obtained by joining all the tuples encoded by \( b_1, \ldots, b_k \) sequentially. Note that the merging function \( \text{mrg} \) is unique if both the encoding function \( \text{enc} \) and the encoding flag \( c \) have been fixed.

**Example 8.** Let \( A \) denote the set of all positive integers and assume that \( A \) is the domain on which we will focus. Let \( P_1, P_2, P_3 \) be three predicates of arities 2, 3, 1, respectively. Now we show how to encode ground positive clauses by elements in \( A \).

Let \( e : A \times A \rightarrow A \) be a function such that

\[
e(m, n) = 2^m + 3^n
\]

for all \( m, n \in A \). It is easy to check that \( e \) is an encoding function on \( A \), and integers 1, 2, 3, 4 are not in the range of \( e \). For \( i \in \{1, 2, 3\} \), fix \( i \) to be the encoding flag for the encodings of positive clauses built from \( P_i \). Then the ground atom \( P_3(1, 3, 5) \) can be encoded by

\[
e(1, 3, 5; 2) = e(e(2, 1, 3), 5) = 2^{155} + 3^5.
\]

Let \( \text{Enc} \) be the encoding flag for encodings of positive clauses. Then the positive clause

\[
P_3(1, 3, 5) \lor P_3(2) \lor P_1(2, 4)
\]

can be encoded by \( e(e(1, 3, 5; 2), e(2; 3), e(2, 4; 1); 4) \).

In classical logic, two positive clauses are equivalent if, and only if, they contain the same set of atoms. Assume that \( c \) is the encoding flag for encodings of positive clauses and \( \text{enc} \) is the encoding function. To capture the equivalence between two positive clauses, some encoding predicates related to \( \text{enc} \) and \( c \) are needed. We define them as follows:

\[
in = \{ (\text{enc}(\vec{a}; c), b) \mid \vec{a} \in A^* \land b \in \text{ELEM}(\vec{a}) \},
\]

\[
\text{subc} = \{ (\text{enc}(\vec{a}; c), \text{enc}(\vec{b}; c)) \mid \vec{a}, \vec{b} \in A^* \land \text{ELEM}(\vec{a}) \subseteq \text{ELEM}(\vec{b}) \},
\]

\[
\text{equ} = \{ (\text{enc}(\vec{a}; c), \text{enc}(\vec{b}; c)) \mid \vec{a}, \vec{b} \in A^* \land \text{ELEM}(\vec{a}) = \text{ELEM}(\vec{b}) \},
\]

where \( \text{ELEM}(\vec{a}) \) and \( \text{ELEM}(\vec{b}) \) denote the sets of elements in \( \vec{a} \) and \( \vec{b} \), respectively. Intuitively, \( \text{in}(a, b) \) asserts that the atom encoded by \( b \) appears in the positive clause encoded by \( a \); \( \text{subc}(a, b) \) asserts that the positive clause encoded by \( a \) is a subclause of that encoded by \( b \); and \( \text{equ}(a, b) \) asserts that the positive clauses encoded by \( a \) and \( b \) are equivalent.

With the above method for encoding, we can now define the translation. Let \( \Pi \) be a disjunctive program. We first construct a class of normal programs related to \( \Pi \) as follows:

1. Let \( C_{\Pi_1} \) denote the set that consists of an individual constant \( c_P \) for each predicate constant \( P \) that occurs in \( \Pi \), and of an individual constant \( c_e \). Here, \( c_e \) will be interpreted as the encoding flag for positive clauses, and \( c_P \) will be interpreted as the encoding flag for atoms built from \( P \). Let \( \Pi_1 \) be the logic program that consists of the rule

\[
\text{Enc}(x, y, c) \rightarrow \bot
\]

for each individual constant \( c \in C_{\Pi_1} \), and the following rules:

\[
\neg\text{Enc}(x, y, z) \rightarrow \text{Enc}(x, y, z)
\]

\[
\neg\text{Enc}(x, y, z) \rightarrow \text{Enc}(x, y, z)
\]

\[
\text{Enc}(x, y, z) \land \text{Enc}(u, v, z) \land \neg x = u \rightarrow \bot
\]

\[
\text{Enc}(x, y, z) \land \text{Enc}(u, v, z) \land \neg y = v \rightarrow \bot
\]
Informally, rules (36)–(38) define that $Enc$ is the graph\(^3\) of a function; rules (34)–(35) state that $Enc$ is injective. Thus, $Enc$ should be the graph of an encoding function. In addition, the rule (31) assures that each $c \in C_\Pi$ is not in the range of $Enc$.

2. Let $\Pi_2$ be the logic program that consists of the following rules:

\[
\begin{align*}
  y = c_\varepsilon & \rightarrow Mrg(x, y, x) \quad (39) \\
  Mrg(x, u, v) \land Enc(u, w, y) \land Enc(v, w, z) & \rightarrow Mrg(x, y, z) \quad (40) \\
  Enc(x, u, y) & \rightarrow In(y, u) \quad (41) \\
  Enc(x, z, y) \land In(x, u) & \rightarrow In(y, u) \quad (42) \\
  x = c_\varepsilon & \rightarrow Subc(x, y) \quad (43) \\
  Subc(u, y) \land Enc(u, v, x) \land In(y, v) & \rightarrow Subc(x, y) \quad (44) \\
  Subc(x, y) \land Subc(y, x) & \rightarrow Equ(x, y) \quad (45)
\end{align*}
\]

Informally, rules (39)–(40) state that $Mrg$ is the graph of the merging function related to $Enc$ and $c_\varepsilon$; rules (41)–(42) implement an inductive version of the definition presented in (28); rules (43)–(44) implement an inductive version of the definition presented in (29); and rules (41)–(45) then assert that $Equ$ is the equivalence relation between positive clauses.

3. Let $\Pi_3$ be the logic program that consists of the rule

\[
True(x) \land Equ(x, y) \rightarrow True(y) \quad (46)
\]

and the rule

\[
\left[\begin{array}{l}
  True(x_1) \land \ldots \land True(x_k) \\
  Enc(y_1, [\theta_1], x_1) \land \ldots \land Enc(y_k, [\theta_k], x_k) \\
  \land Mrg([\gamma_{\bar{u}}], y_1, \ldots, y_k, z) \land \gamma_{\bar{u}}
\end{array}\right] \rightarrow True(z) \quad (47)
\]

for each rule $\gamma \in \Pi$, where:

1. $\theta_1, \ldots, \theta_k$ list all the intensional atoms that have strictly positive occurrences in the body of $\gamma$ for some $k \geq 0$;
2. $\gamma_{\bar{u}}$ is the head of $\gamma$, and $\gamma_{\bar{u}}$ is the conjunction of literals occurring in the body of $\gamma$ but not in the list $\theta_1, \ldots, \theta_k$;
3. $Enc(s_1, [\theta], s_2)$ denotes the formula

\[
u_0^i = c_P \land Enc(u_0^i, t_1, u_1^i) \land \ldots \land Enc(u_{m-1}^i, t_m, u_m^i) \land Enc(s_1, u_m^i, s_2) \quad (48)
\]

if $s_1, s_2$ are two terms, and $\theta$ is an atom of the form $P(t_1, \ldots, t_m)$ for some predicate constant $P$ that occurs in $\Pi$, where each $u_j^i$ is a fresh individual variable;
4. $Mrg([\gamma_{\bar{u}}], y_1, \ldots, y_k, z)$ denotes the formula

\[
Enc(v_0, [\zeta_1], v_1) \land \ldots \land Enc(v_{n-1}, [\zeta_n], v_n) \land Mrg(w_0, y_1, w_1) \land \ldots \\
\land Mrg(w_{k-1}, y_k, w_k) \land v_0 = c_\varepsilon \land w_0 = v_n \land z = w_k \quad (49)
\]

\(^{3}\)Given a $k$-ary function $f$ on some set $A$, the graph of $f$ is defined as the relation $\{(\bar{a}, f(\bar{a})) : \bar{a} \in A^k\}$.\]
if \( \gamma_{\Pi} = \zeta_1 \lor \cdots \lor \zeta_n \) for some atoms \( \zeta_1, \ldots, \zeta_n \) and some integer \( n \geq 0 \).

Intuitively, the rule (46) assures that the progression is closed under the equivalence of positive clauses; the rule (47) simulates the progression operator for the original program. As each positive clause is encoded by an element in the intended domain, the processes of decoding and encoding should be carried out before and after the simulation, respectively.

**Example 9.** Let \( \gamma \) denote the following rule:

\[
P(v) \land \neg Q(v) \rightarrow R(v) \lor S(v)
\]

and suppose \( P, Q, R, S \) are intensional w.r.t. the underlying program. Then, according to the above translation, we can use the following normal rule (which is defined by (47), but with a slight simplification) to simulate the rule \( \gamma \):

\[
\begin{align*}
\text{True}(x_1) \land \text{Enc}(c_P, v, u_1) \land \text{Enc}(y_1, u_1, x_1) \land \\
\text{Enc}(c_R, v, u_2) \land \text{Enc}(c_e, u_2, w_1) \land \text{Enc}(c_S, v, u_3) \land \\
\text{Enc}(w_1, u_3, w_2) \land \text{Merge}(w_2, y_1, z) \land \neg Q(v)
\end{align*}
\]

\rightarrow \text{True}(z). \quad (51)

4. Let \( \Pi_4 \) be the logic program that consists of the rule

\[
x = c_e \rightarrow \text{False}(x)
\]

and the rule

\[
\text{False}(x) \land \text{Enc}(x, [\theta], y) \land \neg \theta \rightarrow \text{False}(y)
\]

for every intensional atom \( \theta \) of the form \( P(\bar{z}_P) \), where \( \bar{z}_P \) is a tuple of pairwise distinct individual variables \( z_1 \cdots z_{k_P} \) that are different from \( x \) and \( y \), and \( k_P \) is the arity of \( P \).

This program is intended to define the predicate \( \text{False} \) as follows: \( \text{False}(a) \) holds in the intended structure if, and only if, \( a \) encodes a positive clause that is false in the structure.

**Remark 2.** Suppose \( A \) is the intended structure. By the definition of \( \Pi_3 \), if \( \text{True}(a) \) is true in \( A \), then \( a \) should be an element in \( A \) that encodes a positive clause, say \( C \), in \( \Gamma_{\Pi}^A \uparrow_\omega \). By the definition of \( \Gamma_{\Pi}^A \), it is not difficult to see that \( A \models C \), which means that \( \text{False}(a) \) will be false in this case. Thus, definitions of \( \text{True} \) and \( \text{False} \) are consistent.

5. Let \( \Pi_5 \) be the logic program consisting of the rule

\[
\text{True}(c_e) \rightarrow \bot
\]

and the following rule

\[
\text{True}(x) \land \text{Enc}(y, [\theta], x) \land \text{False}(y) \rightarrow \theta
\]

for each atom \( \theta \) of the form same as that in \( \Pi_4 \).

Informally, this program asserts that a ground atom is true in the intended structure if, and only if, there is a positive clause containing this atom such that the clause in true and all the other atoms in this clause are false in the intended structure.

Now, we let \( \Pi^\circ \) denote the union of \( \Pi_1, \ldots, \Pi_5 \). This then completes the definition of the translation. The soundness of this translation is assured by the following theorem.

**Theorem 2.** Let \( \Pi \) be a disjunctive program. Then over infinite structures, \( \text{SM}(\Pi) \) is equivalent to \( \exists \pi \text{SM}(\Pi^\circ) \), where \( \pi \) is the set of constants occurring in \( \Pi^\circ \) but not in \( \Pi \).
To prove this result, some notations and lemmas are needed. Let \( v_1 \) and \( \tau \) be the sets of intensional predicates of \( \Pi_1 \) and \( \Pi \), respectively. Let \( \sigma = v_1 \cup v_2 \cup v(\Pi) \). Given a structure \( A \) of \( v(\Pi) \), every encoding expansion of \( A \) is a \( \sigma \)-expansion \( B \) of \( A \) such that

1. \( B \) interprets \( Enc \) as the graph of an encoding function \( enc \) on \( A \) such that no element among \( c^B_\pi \) and \( c^B_P \) (for all \( P \in \tau \)) belongs to the range of \( enc \), and interprets \( Enc \) as the complement of the graph of \( enc \), and interprets \( Defined \) as \( A \times A \);

2. \( B \) interprets \( Mrg \) as the graph of the merging function related to \( enc \) and \( c^B_\pi \), and interprets \( In, Subc, Equ \) as the encoding predicates \( in, subc, equ \) related to \( enc \) and \( c^B_\pi \), respectively.

In the rest of the proof for Theorem 2, unless otherwise mentioned, we assume \( A \) is a structure of \( v(\Pi) \), \( B \) is an encoding expansion of \( A \), and \( enc \) is the encoding function defined by the predicate \( Enc \) in the structure \( B \). Furthermore, we define

\[
[P(a_1, \ldots, a_k)] = enc(a_1, \ldots, a_k; c^B_P),
\]

\[
[\theta_1 \lor \cdots \lor \theta_n] = enc([\theta_1], \ldots, [\theta_n]; c^B_\pi).
\]

Given a set \( \Sigma \) of ground positive clauses, let \([\Sigma]\) be the set of elements \([C]\) for all \( C \in \Sigma \). Let \( \Delta^n(B) \) be the set of elements \( a \in A \) such that \( True(a) \in \Gamma^{\Pi_1 \uparrow n} \).

**Lemma 1.** \( [\Gamma^{\Pi_1 \uparrow \omega}] \subseteq \bigcup_{n \geq 0} \Delta^n(B) \).

**Proof.** \( \subseteq \): By definition, it suffices to show that \( [\Gamma^{\Pi_1 \uparrow n}] \subseteq \Delta^2n(B) \) for all \( n \geq 0 \). We show this by an induction on \( n \). The case for \( n = 0 \) is trivial. Let \( n > 0 \) and assume that \( [\Gamma^{\Pi_1 \uparrow n-1}] \subseteq \Delta^2(n-1)(B) \). Now, our task is to show that \( [\Gamma^{\Pi_1 \uparrow n}] \subseteq \Delta^2n(B) \). Let \( C \in \Gamma^{\Pi_1 \uparrow n} \). By definition, there are a rule \( p_1 \land \cdots \land p_k \rightarrow H \), denoted by \( \gamma_p \), in \( \Pi_1 \), and a sequence of clauses \( C_1 \lor p_1, \ldots, C_k \lor p_k \) in \( \Gamma^{\Pi_1 \uparrow n-1} \) such that \( C \equiv C' \), where \( C' \) denotes the clause \( H \lor C_1 \lor \cdots \lor C_k \). Consequently, there are a rule \( \gamma \) in \( \Pi \) and an assignment \( \alpha \) in \( A \) such that \( \alpha \) satisfies \( \gamma_p \) in \( A \) and that \( \gamma^+ [\alpha] = \gamma_p \). Let \( \alpha \) denote \([C]\), \( a' \) denote \([C']\), and \( a_i \) denote \([C_i \lor p_i]\) for \( 1 \leq i \leq k \). On the other hand, by the inductive assumption, each \( a_i \) should be an element in \( \Delta^2(n-1)(B) \), or equivalently \( True(a_i) \in \Gamma^{\Pi_1 \uparrow 2(n-1)} \); by definition, there exists a rule of the form (47) corresponding to \( \gamma \) in \( \Pi_1 \). Consequently, we have that \( True(a') \in \Gamma^{\Pi_1 \uparrow 2(n-1)} \). As \( Equ(a', a) \) is clearly true in \( B \), according to the rule (46) we then have that \( True(a) \in \Gamma^{\Pi_1 \uparrow 2n} \), which implies \( a \in \Delta^2n(B) \).

\( \supseteq \): It suffices to show that \( [\Gamma^{\Pi_1 \uparrow n}] \supseteq \Delta^n(B) \) for all \( n \geq 0 \). Similarly, we show it by an induction on \( n \). The case for \( n = 0 \) is trivial. Let \( n > 0 \) and assume that \( [\Gamma^{\Pi_1 \uparrow n-1}] \supseteq \Delta^n(B) \). Let \( a \in \Delta^n(B) \). We then have \( True(a) \in \Gamma^{\Pi_1 \uparrow n} \). By definition, \( True(a) \) must be generated by either rule (46) or rule (47). If the first case is true, there must exist an element \( b \in \Gamma^{\Pi_1 \uparrow n-1} \) such that \( Equ(a, b) \) is true in \( B \). By the inductive assumption, there is a positive clause \( C \) such that \( [C] = b \) and \( C \in \Gamma^{\Pi_1 \uparrow n-1} \). By the definition of \( equ \), it is also clear that \( a \) encodes a clause \( C_0 \) that contains exactly the set of atoms in \( C \). By the definition of the progression operator, it is clear that \( C_0 \in \Gamma^{\Pi_1 \uparrow n-1} \subseteq \Gamma^{\Pi_1 \uparrow n} \).

Now, it remains to consider the case that \( True(a) \) is generated by the rule (47). By definition, there should be (i) a sequence of elements \( b_1, \ldots, b_k \in B \) such that \( True(b_i) \in \Gamma^{\Pi_1 \uparrow n-1} \) for \( 1 \leq i \leq k \), (ii) a rule \( \gamma \in \Pi \) such that \( \theta_1, \ldots, \theta_k \) are the set of intensional atoms positively appearing in the body, (iii) a sequence of elements \( a_1, \ldots, a_k \in B \), and (iv) an assignment \( \alpha \) in \( B \) such that \( Mrg([H], a_1, \ldots, a_k, a) \) and all \( Enc(a_i, [p_i], b_i) \) are true in \( B \), where \( p_i = \theta_i[\alpha] \) and \( H \) denotes the clause \( \gamma_\Pi[\alpha] \). By the inductive assumption,
there exist a sequence of clauses \( C_1 \lor p_1, \ldots, C_k \lor p_k \) in \( \Gamma_{\Pi}^\uparrow_{\neg\pi-1} \) such that \( [C_i \lor p_i] = b_i \) and \( [C_i] = a_i \). Let \( C \) denote the clause \( H \lor C_1 \lor \cdots \lor C_k \). By definition, \( C \) should be in \( \Gamma_{\Pi}^\uparrow_{\neg\pi} \). It is also clear that \( a = [C] \), which implies \( a \in [\Gamma_{\Pi}^\uparrow_{\neg\pi}] \) as desired. \( \square \)

Again, unless otherwise mentioned, let us fix \( C \) as an \( \upsilon(\Pi^\omega) \)-expansion of \( B \) that interprets \( \text{True} \) as the set \( [\Gamma_{\Pi}^\uparrow_{\neg\omega}] \), and that interprets the predicate \( \text{False} \) as the set
\[
\{[C] \mid C \in \text{GPC}(\tau, A) \& \text{INS}(A, \tau) \models \neg C\} \quad (58)
\]
where \( \text{GPC}(\tau, A) \) denotes the set of all ground positive clauses built from predicates in \( \tau \) and elements in \( A \). Such a structure is also called a progression expansion of \( A \).

**Lemma 2.** \( \text{INS}(A, \tau) \) is a minimal model of \( \Gamma_{\Pi}^\uparrow_{\neg\omega} \) iff \( \text{INS}(C, \tau) \) is a minimal model of \( \Pi^C_5 \).

Roughly speaking, the soundness of Lemma 2 is assured by the result that every head-cycle-free disjunctive program is equivalent to a normal program obtained by shifting [Ben-Eliyahu and Dechter 1994]. Note that every set of positive clauses is head-cycle-free, and \( \Pi_4 \) and \( \Pi_5 \) are designed for the simulation of shifting. Now, let us present the proof.

**Proof of Lemma 2.** We only show the direction \( \Longrightarrow \). The converse can be proved by a routine check in a similar way. Assume that \( \text{INS}(A, \tau) \) is a minimal model of \( \Gamma_{\Pi}^\uparrow_{\neg\omega} \).

We first show that \( \text{INS}(C, \tau) \) is a model of \( \Pi^C_5 \). Let \( B \) denote the restriction of \( C \) to \( \tau \). It is clear that \( B \) is an encoding expansion of \( A \). By assumption, \( \text{INS}(A, \tau) \) is a model of \( \Gamma_{\Pi}^\uparrow_{\neg\omega} \).

So, we must have that \( \bot \notin [\Gamma_{\Pi}^\uparrow_{\neg\omega}] \). By Lemma 1, the element \( \epsilon^C_\gamma \), which encodes the empty clause \( \bot \), is then not in \( \Delta^\alpha(\Pi^C_5) \). Equivalently, \( \text{True}(c_\alpha) \) is false in \( C \). This means that the rule (54) is satisfied by \( C \). Let \( \gamma \) be any rule of the form (55) and \( \alpha \) an arbitrary assignment in \( C \) such that \( \alpha \) satisfies the body of \( \gamma \) in \( C \). By definition, there should exist a clause \( C \in [\Gamma_{\Pi}^\uparrow_{\neg\omega}] \) such that \( \alpha(x) = [C] \). As \( \text{Enc}(y, [\theta], x) \) is satisfied by \( \alpha \) in \( C \), by definition there exists a subclause \( C_0 \) of \( C \) such that \( \alpha(y) = [C_0] \) and \( C = C_0 \lor p \) where \( p \) denotes \( \theta[x] \). Since \( \alpha \) also satisfies \( \text{False}(y) \) in \( C \) and \( C \) interprets \( \text{False} \) as the relation (58), we conclude that \( \text{INS}(A, \tau) \models \neg C_0 \). On the other hand, we have \( \text{INS}(A, \tau) \cap C \neq \emptyset \) because \( \text{INS}(A, \tau) \) satisfies \( C \). Combining these conclusions, it must hold that \( \text{INS}(C, \tau) \) satisfies \( \text{INS}(A, \tau) \), which implies \( p \in \text{INS}(C, \tau) \) immediately. Thus, \( \text{INS}(C, \tau) \) is indeed a model of \( \Pi^C_5 \).

Let \( \pi \) be the set of intensional propositional constants of \( \Pi^C_5 \). Clearly, each intensional propositional constant of \( \Pi^A_\pi \) is in \( \pi \). Next, we want to show that \( \text{INS}(C, \tau) \) is a \( \pi \)-minimal model of \( \Pi^C_5 \). Let \( M \) be any model of \( \Pi^C_5 \) such that \( M \subseteq \text{INS}(C, \tau) \) and \( M \setminus \pi = \text{INS}(C, \tau) \setminus \pi \). It suffices to show that \( M \cap \pi \supseteq \text{INS}(C, \tau) \cap \pi \). We claim that for each atom \( p \in \pi \cap \text{INS}(A, \tau) \) there exists at least one clause, say \( C_p \), in \( [\Gamma_{\Pi}^\uparrow_{\neg\omega}] \) such that \( C_p \cap \text{INS}(A, \tau) = \{p\} \). Otherwise, let \( N = \text{INS}(A, \tau) \setminus \{p\} \); it is obvious that \( N \) is also a model of \( [\Gamma_{\Pi}^\uparrow_{\neg\omega}] \), a contradiction. With this claim, for each atom \( p \in \pi \cap \text{INS}(C, \tau) \) there should be a rule \( \gamma \) of the form (55) and an assignment \( \alpha \) in \( C \) such that \( \theta[\alpha] = p \), \( \alpha(x) = [C_p] \) and \( C \models \text{Enc}(y, [\theta], x)[\alpha] \). It is clear that \( \gamma[\alpha] \in \Pi^C_5 \). Let \( C_0 \) be the clause obtained from \( C_p \) by removing \( p \). By the definition of \( C_p \), it is clear that each atom in \( C_0 \) should be false in \( A \) (so it is false in \( C \) too). As \( \alpha(y) \) encodes \( C_0 \), the ground atom \( \text{False}(y)[\alpha] \) should be in \( \text{INS}(C, \tau) \setminus \pi = M \setminus \pi \). It is also easy to check that \( \text{True}(x)[\alpha], \text{Enc}(y, [\theta], x)[\alpha] \in \text{INS}(C, \tau) \setminus \pi = M \setminus \pi \). Since \( M \) satisfies \( \gamma[\alpha] \), we must have \( p = \theta[\alpha] \in M \). Thus, we can conclude that \( M \cap \pi \supseteq \text{INS}(C, \tau) \cap \pi \). \( \square \)

With these lemmas, we can then prove Theorem 2.
PROOF OF THEOREM 2. By Proposition 2, it suffices to show that

$$\text{SM}(\Pi) \equiv_{\text{INF}} \exists \pi(\text{SM}(\Pi_1) \land \cdots \land \text{SM}(\Pi_5)).$$  \hspace{1cm} (59)

\text{“⇒”}: Let \(A\) be an infinite model of \(\text{SM}(\Pi)\), and \(B\) be an encoding expansion of \(A\). As \(A\) is infinite, such an expansion always exists. It is easy to check that \(B\) is a stable model of both \(\Pi_1\) and \(\Pi_2\). Let \(C\) be the progression expansion of \(A\) that is also an expansion of \(B\). By Theorem 1, \(\text{INS}(B, \tau) = \text{INS}(A, \tau)\) should be a minimal model of \(\Pi_3 \uparrow_\omega\). By Lemma 1 and definition, \(\text{INS}(B, \tau)\) is also a minimal model of \(\Pi_3 \uparrow_\omega\). By Theorem 1 again, \(B\) is then a stable model of \(\Pi_3\), which implies that so is \(C\). It is also easy to check that \(C\) is a stable model of \(\Pi_4\). On the other hand, since \(\text{INS}(A, \tau)\) is a minimal model of \(\Pi_4 \uparrow_\omega\), by Lemma 2, \(\text{INS}(C, \tau)\) should be a minimal model of \(\Pi_5^C\), which means that \(C\) is a stable model of \(\Pi_5\) by Proposition 1. Thus, \(A\) is a model of the right-hand side of (59).

\text{“⇐”:} Let \(A\) be an infinite model of the right-hand side of (59). Then there exists an \(\nu(\Pi)\)-expansion \(C\) of \(A\) such that \(C\) satisfies \(\text{SM}(\Pi_i)\) for all \(i \in \{1, \ldots, 5\}\). Let \(B\) be the restrictions of \(C\) to \(\sigma\). Then, by a routine check, it is easy to show that \(B\) is an encoding expansion of \(A\). As \(C\) is a stable model of \(\Pi_3\), by Theorem 1, \(\text{INS}(C, \nu_3)\) is then a minimal model of \(\Pi_3 \uparrow_\omega = \Pi_3 \uparrow_\omega\). Furthermore, by Lemma 1 and the conclusion that \(C\) satisfies \(\text{SM}(\Pi_4)\), we then have that \(C\) is a progression expansion of \(A\). On the other hand, since \(C\) is also a stable model of \(\Pi_5\), by Proposition 1 we can conclude that \(\text{INS}(C, \tau)\) is a minimal model of \(\Pi_5^C\). Thus, by Lemma 2 we immediately have that \(\text{INS}(A, \tau)\) is a minimal model of \(\Pi_5 \uparrow_\omega\). By Theorem 1, \(A\) must be a stable model of \(\Pi\), which completes the proof. \(\square\)

REMARK 3. Note that, given any finite domain \(A\), there is no injective function from \(A \times A\) into \(A\). Thus, we cannot expect that the above translation works on finite structures.

COROLLARY 1. DLP \(\simeq_{\text{INF}}\) NLP.

Now, let us focus on the relationship between logic programs and second-order logic. The following proposition says that, over infinite structures, normal programs are more expressive than the existential second-order logic, which then strengthens a result in [Asuncion et al. 2012] where such a separation over arbitrary structures was obtained.

PROPOSITION 3. NLP \(\not\subseteq_{\text{INF}}\) \(\Sigma_1^1\).

To show this, our main idea is to find a property that can be defined by a normal program but not by any existential second-order sentence. A property that satisfies the mentioned conditions is defined as follows. Let \(\nu_k\) be the vocabulary consisting of a binary predicate \(E\) and two individual constants \(c\) and \(d\). Let \(\text{Reach}_i\) be the class of infinite \(\nu_k\)-structures in each of which there is a finite path from \(c\) to \(d\) via edges in \(E\). Now, we show the result.

PROOF OF PROPOSITION 3. We first show that \(\text{Reach}_i\) is definable in NLP over infinite structures. Let \(\Pi\) be a normal program that consists of the following rules:

\[ P(c), \]  \hspace{1cm} (60)
\[ P(x) \land E(x, y) \rightarrow P(y), \]  \hspace{1cm} (61)
\[ \neg P(d) \rightarrow \bot. \]  \hspace{1cm} (62)

It is easy to see that \(\exists \text{FSM}(\Pi)\) defines the property \(\text{Reach}_i\) over infinite structures.

Next, we prove that \(\text{Reach}_i\) is undefinable in \(\Sigma_1^1\) over infinite structures. Towards a contradiction, assume that there is a first-order sentence \(\varphi\) and a finite set \(\tau\) of predicates
such that $\exists \tau \varphi$ is in $\Sigma_1^1$ and defines $\text{Reach}_i$, over infinite structures. Let $R$ be a binary predicate that has no occurrence in $\tau$, and let $\psi$ denote the formula
\[
\forall x \exists y R(x, y) \land \forall x \neg R(x, x) \land \forall x \forall y \forall z (R(x, y) \land R(y, z) \rightarrow R(x, z)).
\]
Intuitively, it asserts that the relation $R$ is transitive and irreflexive, and each element in the domain has a successor w.r.t. $R$. It is obvious that such a relation exists iff the domain is infinite. Thus, the formula $\exists \tau \varphi \land \exists R \psi$ defines $\text{Reach}_i$ over arbitrary structures.

Moreover, let $\gamma_0(x, y)$ be $x = y$; for $n > 0$ let $\gamma_n(x, y)$ denote the formula
\[
\exists z_n (\gamma_{n-1}(x, z_n) \land E(z_n, y)),
\]
where each $\gamma_n(x, y)$ asserts that there is a path of length $n$ from $x$ to $y$. Let $\Lambda$ be the set of sentences $\neg \gamma_n(c, d)$ for all $n \geq 0$. Now we prove a property as follows.

**Claim.** $\Lambda \cup \{\exists \tau \varphi, \exists R \psi\}$ is satisfiable.

To show this, it suffices to show that the first-order theory $\Lambda \cup \{\varphi, \psi\}$ is satisfiable. Let $\Phi$ be a finite subset of $\Lambda$, and let $n = \max \{m \mid \neg \gamma_m(c, d) \in \Phi\}$. Let $A$ be an infinite model of $\psi$ with vocabulary $v(\varphi) \cup v(\psi)$ in which the minimal length of paths from $c$ to $d$ via edges in $E$ is an integer $> n$. Then $A$ is clearly a model of $\Phi \cup \{\varphi, \psi\}$. Due to the arbitrariness of $\Phi$, by the compactness we then have the desired claim.

Let $A$ be any model of $\Lambda \cup \{\exists \tau \varphi, \exists R \psi\}$. Then according to $\exists R \psi$, $A$ should be infinite, and by $\Lambda$, there is no path from $c$ to $d$ via $E$ in $A$. However, according to $\exists \tau \varphi$, every infinite model of it should be $c$-to-$d$ reachable, a contradiction. Thus, the property $\text{Reach}_i$ is then undefinable in $\Sigma_1^1$ over infinite structures. This completes the proof immediately.

The following separation immediately follows from the proof of Theorem 4.1 in [Eiter et al. 1996]. Although their statement refers to the whole class of arbitrary structures, the proof still works if only infinite structures are considered.

**Proposition 4.** $\Sigma_2^1 \not\equiv_{\text{INF}} \text{DLP}.$

### 4. FINITE STRUCTURES

In this section we focus on the expressiveness of logic programs over finite structures. We first consider the relationship between disjunctive and normal programs. Unfortunately, in the general case, we have the following result. Note that the direction “only if” had been also showed by Asuncion et al. [2012].

**Proposition 5.** $\text{DLP} \equiv_{\text{FIN}} \text{NLP}$ iff $\text{NP} = \text{coNP}.$

**Proof.** By Fagin’s Theorem [Fagin 1974] and Stockmeyer’s logical characterization of the polynomial-time hierarchy [Stockmeyer 1977], we have that $\Sigma_2^1 \equiv_{\text{FIN}} \Sigma_1^1$ iff $\Sigma_0^P = \text{NP}$. By a routine complexity-theoretic argument, it is also true that $\Sigma_2^P = \text{NP}$ iff $\text{NP} = \text{coNP}$. On the other hand, according to the proof of Theorem 6.3 in [Eiter et al. 1997], or by Proposition 7 in this section, Leivant’s normal form [Leivant 1989] and the definition

---

4 The proposition was first presented in an earlier version of this paper [Zhang and Zhang 2013a]. Recently, such an equivalence was also observed by Zhou [2015]. For traditional logic programs under the query equivalence, a similar result follows from the expressiveness results proved by [Eiter et al. 1997; Schlipf 1995].

5 In their characterizations of complexity classes, no function constant of positive arity is allowed. However, this restriction can be removed since functions can be easily simulated by predicates and first-order quantifiers.
of SM, we can conclude that \( DLP \simeq_{\text{FIN}} \Sigma_1^2 \). By Proposition 6 in this section, it holds that \( NLP \simeq_{\text{FIN}} \Sigma_1 \). Combining these conclusions, we then have the desired proposition. \( \square \)

This result shows how difficult it is to separate normal programs from disjunctive programs over finite structures. To know more about the relationship, we will try to prove a weaker separation between these two classes. To achieve this goal, we need to study the relationship between logic programs and second-order logic.

For the class of normal programs, we have the following characterization:

**Proposition 6.** \( \text{NLP}^F_n \simeq_{\text{FIN}} \Sigma_{1,n}^F[\forall^*] \) for all \( n > 1 \).

To prove it, we have to develop a translation that turns normal programs to first-order sentences. The main idea is to extend the Clark completion by simulating the progression.

Now, let us give the translation. Let \( \Pi \) be a normal program and \( n \) be the maximum arity of intensional predicates of \( \Pi \). Without loss of generality, assume the head of every rule in \( \Pi \) is of the form \( P(\bar{x}) \), where \( P \) is a \( k \)-ary predicate for some \( k \geq 0 \), and \( \bar{x} \) is the tuple of \( k \) distinct individual variables \( x_1, \ldots, x_k \). Let \( \prec \) be a new binary predicate and \( \varpi \) a universal first-order sentence asserting that \( \prec \) is a strict linear order. Given two tuples \( \bar{s}, \bar{t} \) of terms of the same length, let \( \bar{s} \prec \bar{t} \) be a quantifier-free formula asserting that \( \bar{s} \) is less than \( \bar{t} \) w.r.t. the lexicographic order extended from \( \prec \) naturally. For example, suppose \( \bar{s} \) and \( \bar{t} \) denote the binary tuples \( (s_1, s_2) \) and \( (t_1, t_2) \), respectively; then, to express that \( \bar{s} \) is less than \( \bar{t} \) w.r.t. the extended strict linear order can be expressed by the following quantifier-free formula:

\[
\bar{s}_1 < t_1 \lor (s_1 = t_1 \land s_2 < t_2).
\]

Let \( \tau \) be the set of intensional predicates of \( \Pi \). Let \( c \) be the least integer \( \geq \log_2 |\tau| + n \). We fix \( P \) to be a \( k \)-ary predicate in \( \tau \) and let \( \lambda = P(x_1, \ldots, x_k) \). Suppose \( \gamma_1, \ldots, \gamma_l \) list all the rules in \( \Pi \) whose heads are \( \lambda \), and suppose each \( \gamma_i \) is of the form

\[
\eta^i \land \theta^i_1 \land \cdots \land \theta^i_{m_i} \rightarrow \lambda
\]

where \( \theta^i_1, \ldots, \theta^i_{m_i} \) list all the positive intensional conjuncts in the body of \( \gamma_i \), \( \eta^i \) is the conjunction of other conjuncts that occur in the body of \( \gamma_i \), \( m_i \geq 0 \), and \( \bar{y}_i \) is the tuple of all individual variables that occur in \( \gamma_i \) but not in \( \lambda \).

Next, let \( \psi_P \) denote the following first-order sentence:

\[
\forall x_1 \cdots \forall x_k \left[ \lambda \rightarrow \bigvee_{i=1}^{l} \exists \bar{y}_i \left[ \eta^i \land \bigwedge_{j=1}^{m_i} (\theta^i_j \land D\text{Less}(\theta^i_j, \lambda)) \right] \right]
\]

where, for all intensional atoms \( \theta \) and \( \theta_0 \), let \( \text{ORD}(\theta) \) denote the tuple \((a^Q_O(\bar{i}), \ldots, a^Q_O(\bar{i}))\) if \( \theta \) is of the form \( Q(\bar{i}) \); let \( a^Q_O \) be a fresh function constant whose arity is the same as that of \( Q \); and let \( D\text{Less}(\theta, \theta_0) \) denote the quantifier-free formula \( \text{ORD}(\theta) \prec \text{ORD}(\theta_0) \).

Now we define \( \varphi_{\Pi} \) as a conjunction of the sentences \( \varpi \) and \( \Pi \), and \( \psi_{P} \) for all \( P \in \tau \). Let \( \sigma \) denote the set of function constants \( a^Q_S \) for all \( Q \in \tau \) and all \( s \in \{1, \ldots, c\} \). Clearly, \( \exists \sigma \varphi_{\Pi} \) is equivalent to a sentence in \( \Sigma_{1,n}^F[\forall^*] \) by introducing Skolem functions if \( n > 1 \).

**Example 10 (Example 5 continued).** Let \( \Pi \) denote the following normal program which is obtained from the program presented in Example 5 by a slight modification:

\[
y = a \rightarrow P(y), \tag{68}
\]

\[
P(x) \land E(x, y) \rightarrow P(y), \tag{69}
\]
\[ y = b \land \neg P(b) \rightarrow P(y). \]  

(70)

Clearly, the only intensional predicate constant is \( P \). Let \( \tau = \{ P \} \). Then we have that

\[ c = \log_2 |\tau| + n = 0 + 1 = 1, \]

(71)

where \( n \) denotes the maximum arity of predicate constants that occur in \( \tau \). According to the translation defined above, \( \psi_P \) is the following first-order formula:

\[ \forall y(P(y) \rightarrow y = a \lor \exists x(E(x, y) \land P(x) \land a^P(x) \prec o_P^1(y)) \lor (y = b \land \neg P(b))). \]

(72)

where \( o_P^1 \) is a fresh unary function. Moreover, \( \varphi_{11} \) is the formula \( \exists \land \Pi \land \psi_P \). It is clear that \( \exists o_P^1 \varphi_{11} \) is equivalent to the \( \Sigma_{1,n}^1 \)-sentence \( \exists o_P^1 \exists f(\varphi \land \Pi \land \vartheta) \), where \( \vartheta \) denotes

\[ \forall y(P(y) \rightarrow y = a \lor (E(f(y), y) \land P(f(y)) \land o_P^1(f(y)) \prec o_P^1(y)) \lor (y = b \land \neg P(b))). \]

(73)

and \( f \) is a unary Skolem function introduced to eliminate the existential quantifier \( \exists x \).

Next, let us show the soundness of the presented translation:

**Lemma 3.** Given any finite structure \( A \) of \( v(\Pi) \) with at least two elements in the domain, we have \( A \models SM(\Pi) \) iff \( A \models \exists o \varphi_{11} \).

**Proof.** "\( \Longrightarrow \)" Let \( A \) be a finite stable model of \( \Pi \) with at least two elements in the domain. Let \( N \) be the cardinality of \( A \). Without loss of generality, we assume \( A \) is the set of natural numbers \( < N \). Note that every finite structure with domain size \( N \) is isomorphic to a structure with this domain. Given any ground intensional atom \( p \), we define

\[ \ell(p) = \max\{m < N^\omega \mid p \notin \Gamma_{11}^A \uparrow_m \}. \]

(74)

Let \( B \) be an \( v(\varphi_{11}) \)-expansion of \( A \) in which

1. the predicate constant \( \prec \) is interpreted as the relation \( \{(a, b) \in A \times A \mid a < b\} \);
2. for each predicate constant \( P \in \tau \) of arity \( s \geq 0 \) and each integer \( i \in \{1, \ldots, c\} \), the function constant \( o_P^i \) is interpreted as a function \( g \) of arity \( s \) such that \( g(\bar{a}) = d_i \) for all \( s \)-tuples \( \bar{a} \) on \( A \) if \( (d_1, \ldots, d_s) \) is the representation of \( \ell(P(\bar{a})) \) in the base-\( N \) numeral system. (For example, suppose \( N = 8, c = 3 \) and \( \ell(P(\bar{a})) = 22 \); then the desired representation of \( \ell(P(\bar{a})) \) in the base-8 numeral system is \( (0, 2, 6) \).

By a routine check, one can show that \( B \) is a model of \( \varphi_{11} \). From this, we know that \( A \) is indeed a model of \( \exists o \varphi_{11} \), which proves the desired direction.

"\( \Longleftarrow \)" Let \( A \) be a finite model of \( \exists o \varphi_{11} \). We want to show that \( A \) is a stable model of \( \Pi \). By Theorem 1, it suffices to show that \( \text{Ins}(A, \tau) = \Gamma_{11}^A \uparrow_m \). Clearly, there exists a model \( B \) of \( \varphi_{11} \) that is an \( v(\varphi_{11}) \)-expansion of \( A \). By the formula \( \varphi \), we know that \( B \) must interpret \( \prec \) as a strict linear order on \( B \). We first show that \( (\ast) \Gamma_{11}^A \uparrow_m \subseteq \text{Ins}(A, \tau) \) for all \( m \geq 0 \), which will be done by an induction on \( m \). The case of \( m = 0 \) is trivial. Let \( m > 0 \) and assume that \( \Gamma_{11}^A \uparrow_{m-1} \subseteq \text{Ins}(A, \tau) \). Our task is to show \( \Gamma_{11}^A \uparrow_m \subseteq \text{Ins}(A, \tau) \). Let \( p \) be a ground atom in \( \Gamma_{11}^A \uparrow_m \). By definition, there exist a rule \( \gamma_i \in \Pi \) of the form

\[ \eta^i \land \theta^i_1 \land \cdots \land \theta^i_{m_i} \rightarrow \lambda \]

(75)

and an assignment \( \alpha \) in \( A \) such that (i) \( \lambda[\alpha] = p \), (ii) \( \alpha \) satisfies \( \eta^i \) in \( A \) (so equivalently, in \( B \)), and (iii) for each atom \( \theta^j_i \), it holds that \( \theta^j_i[\alpha] \in \Gamma_{11}^A \uparrow_{m-1} \). By the inductive assumption,
we have \( \theta_j^i[\alpha] \in \text{INS}(A, \tau) \), which means that \( B \models \theta_j^i[\alpha] \). As \( \alpha \) clearly satisfies the rule \( \gamma_i \) in \( B \), we conclude that \( \alpha \) satisfies \( \lambda \) in \( B \), which implies that

\[
p = \lambda[\alpha] \in \text{INS}(B, \tau) = \text{INS}(A, \tau).
\]

(76)

So, the claim (\#) is true. From it, we then have that \( \Gamma_i^A \uparrow_\omega \subseteq \text{INS}(A, \tau) \) as desired.

Now, it remains to show that \( \text{INS}(A, \tau) \subseteq \Gamma_i^A \uparrow_\omega \). Towards a contradiction, assume that it is not true. We then have that \( \Gamma_i^A \uparrow_\omega \subseteq \text{INS}(A, \tau) \) by the previous conclusion. Given any two ground atoms \( p_1, p_2 \in \text{INS}(A, \tau) \), we write \( p_1 < p_2 \) if \( \text{LESS}(p_1, p_2) \) is true in \( B \). Let \( p \) be a \( \prec \)-minimal atom in \( \text{INS}(A, \tau) \setminus \Gamma_i^A \uparrow_\omega \) and suppose \( p = P(\bar{d}) \) for some \( P \in \tau \). Let \( \alpha \) be an assignment in \( B \) such that \( \alpha(\bar{x}) = \bar{d} \). By definition, \( \alpha \) should satisfy \( \psi_p \) (in which \( \lambda[\alpha] = p \)) in \( B \). So, there are an integer \( i \in \{1, \ldots, l\} \) and an assignment \( \alpha_0 \in B \) such that (i) \( \alpha_0(\bar{x}) = \bar{d} \), (ii) \( \eta_i^i[\alpha_0] \) is true in \( B \), and (iii) for all \( j, q_j \in \text{INS}(B, \tau) = \text{INS}(A, \tau) \) and \( q_j < \lambda[\alpha_0] \), where \( q_j \) denotes \( \theta_j^i[\alpha_0] \). As \( \lambda[\alpha_0] = \lambda[\alpha] = p \) and \( p \) is \( \prec \)-minimal in \( \text{INS}(A, \tau) \setminus \Gamma_i^A \uparrow_\omega \), we conclude \( q_j \in \Gamma_i^A \uparrow_\omega \) for all \( j \). According to the definition of \( \psi_p \), the rule \( \gamma_i \) (of the form (66)) is in \( \Pi \), which implies that

\[
\gamma_i^+[\alpha_0] = q_1 \land \cdots \land q_m \rightarrow p \in \Pi^A.
\]

(77)

By definition, we have that \( p \in \Gamma_i^A \uparrow_\omega \), a contradiction as desired. \( \square \)

**Remark 4.** Let \( m \) and \( n \) be the number and the maximal arity of intensional predicates respectively. The maximal arity of auxiliary constants in our translation is only \( n \) (that of the ordered completion in [Asuncion et al. 2012] is \( 2n \)), which is optimal if Conjecture 1 in [Durand et al. 2004] is true. Moreover, the number of auxiliary constants in our translation is \( m \cdot (\lceil \log_2 m \rceil + n) \), while that of the ordered completion is \( m^2 \).

**Remark 5.** Similar to the work in [Asuncion et al. 2012], one can develop an answer set solver by calling an SMT solver. By the comparison in the above remark, this approach is rather promising. In addition, as a strict partial order is available in almost all the SMT solvers (e.g., built-in arithmetic relations), our translation can be easily optimized.

Now we are in the position to prove Proposition 6.

**Proof of Proposition 6.** “\( \geq_{\text{FIN}} \)”: Let \( \varphi \) be any sentence in \( \Sigma_{1,n}^{\text{FIN}}[\forall^*] \). It is obvious that \( \varphi \) can be written as an equivalent sentence of the form \( \exists r \forall \psi(e_i \land \cdots \land e_k) \) for some \( k \geq 0 \), where each \( e_i \) is a disjunction of literals, i.e., atoms or negated atoms, and \( r \) a finite set of functions or predicates of arity \( \leq n \). Let \( \Pi \) be a logic program consisting of the rule \( \tilde{e}_i \rightarrow \bot \) for each \( i \in \{1, \ldots, k\} \), where \( \tilde{e}_i \) is obtained from \( e_i \) by substituting \( \tilde{e} \) for each negated atom \( \neg \theta \), followed by substituting \( \neg \theta \) for each atom \( \theta \), and finally by substituting \( \land \) for \( \lor \). It is easy to check that \( \exists r \text{SM}(\Pi) \) is in \( \text{NLP}_n^{\varphi} \) and equivalent to \( \varphi \).

“\( \leq_{\text{FIN}} \)”: Let \( C = 1 \) (respectively, \( C = 2 \)) denote the class of finite structures with exactly one (respectively, at least two) element(s) in the domain. Let \( \Pi \) be a normal program and \( \tau \) a finite set of predicates and functions such that \( \exists r \text{SM}(\Pi) \) is in \( \text{NLP}^{\varphi} \). It is trivial to construct a sentence, say \( \zeta \), in \( \Sigma_{1,n}^{\text{FIN}}[\forall^*] \) such that \( \exists r \text{SM}(\Pi) \) is equivalent to \( \zeta \) over \( C = 1 \). (Note that, if the domain is a singleton, a first-order logic program will regress to a propositional one.) By Lemma 3, there is also a sentence \( \psi \) in \( \Sigma_{1,n}^{\text{FIN}}[\forall^*] \) such that \( \exists r \text{SM}(\Pi) \)

\(^{4}\text{Conjecture 1 in [Durand et al. 2004] implies that ESO}_{n}^{\varphi} \geq_{\text{FIN}} \text{ESO}_{n}^{\varphi}, \text{where ESO}_{n}^{\varphi} \text{ denotes the class of sentences in ESO}_{n}^{\varphi} \text{ that involve at most } n \text{ individual variables.}
is equivalent to $\psi$ over $C^{>1}$. Let $\varphi$ denote the following sentence:
\[
(\exists x \forall y (x = y) \land \zeta) \lor (\exists x \exists z (\neg x = z) \land \psi).
\]
Informally, the formula $\varphi$ first tests whether the intended domain is a singleton or not. $\zeta$ will be activated if the answer is true, and $\psi$ will be activated otherwise. Thus, it is easy to show that $\exists \sigma SM(\Pi)$ is equivalent to $\varphi$ over finite structures. It is also clear that $\varphi$ can be written to be an equivalent sentence in $\Sigma^{1}_{1,n}[\forall^{*}]$. Note that every first-order quantifier can be regarded as a second-order quantifier over a function variable of arity 0. \hfill \square

**Remark 6.** Again, if one assumes Conjecture 1 in [Durand et al. 2004], by the main result of [Grandjean 1985], $\text{NLP}_k^\Pi$ then exactly captures the class of languages computable in $O(n^k)$-time (where $n$ denotes the size of input) in Nondeterministic Random Access Machines (NRAMs), which implies that whether an extensional database can be expanded to a stable model of a disjunctive program can be checked in $O(n^k)$-time in an NRAM.

By Proposition 6 and the fact that functions can be simulated by introducing auxiliary predicates in both logic programs and second-order logic, we have the following result:

**Corollary 2.** $\text{NLP} \simeq_{\text{FIN}} \Sigma^{1}_{1}.$

To establish the mentioned separation, we still need to investigate the translatability from a fragment of second-order logic to disjunctive programs. For convenience, in the rest of this paper, we fix $\text{Succ}$ as a binary predicate, $\text{First}$ and $\text{Last}$ as two unary predicates, and $\nu_5$ as the set that consists of these predicates. Unless otherwise mentioned, every logic program or formula to be considered is always assumed to contain no predicate from $\nu_5$.

A structure $A$ is called a successor structure if all of the following hold:

1. the vocabulary of $A$ contains all the predicates in $\nu_5$, and
2. $\text{Succ}^A$ is a binary relation $S$ on $A$ such that the transitive closure of $S$ is a strict linear order, and that $|\{(b|(a,b) \in S)\}| \leq 1$ and $|\{(b| (b,a) \in S)\}| \leq 1$ for all $a \in A$, and
3. $\text{First}^A$ (respectively, $\text{Last}^A$) consists of the least element (respectively, the largest element) in $A$ w.r.t. the strict linear order defined by $\text{Succ}^A$.

Note that, by definition, both the least and largest elements must exist in a successor structure. This means that every successor structure is always finite.

Let $\text{SUC}$ denote the class of successor structures, and let $\Sigma^{1}_{2,n}[\forall^n \exists^*]$ denote the class of sentences in $\Sigma^{1}_{2,n}[\forall^n \exists^*]$ that involve at most $n$ universal quantifiers. Now let us show that:

**Lemma 4.** $\Sigma^{1}_{2,n}[\forall^n \exists^*] \leq_{\text{SUC}} \text{DLP}_n$ for all $n > 0$.

**Proof.** Fix $n > 0$, and let $\exists \forall \sigma \varphi$ be a sentence in $\Sigma^{1}_{2,n}[\forall^n \exists^*]$ where $\tau, \sigma$ are finite sets of predicates of arities $\leq n$. It suffices to show that there are a disjunctive program $\Pi$ and a set $\pi$ of auxiliary predicates such that $\forall \sigma \varphi \equiv_{\text{SUC}} \exists \pi SM(\Pi)$ and that $\exists \pi SM(\Pi)$ belongs to $\text{DLP}_n$. Without loss of generality, let us assume that $\varphi$ is of the form
\[
\forall \vec{x} \exists \vec{y} (\vartheta_1(\vec{x}, \vec{y}) \lor \cdots \lor \vartheta_m(\vec{x}, \vec{y})),
\]
where $m \geq 0$, each $\vartheta_i$ is a finite conjunction of literals, and the length of $\vec{x}$ is exactly $n$.

Before presenting the program $\Pi$, let us first define some notations. Suppose $\vec{u}$ and $\vec{v}$ are two tuples of individual variables $u_1 \cdots u_k$ and $v_1 \cdots v_k$, respectively. Let $\text{First}(\vec{u})$ denote
the conjunction of First(\(u_i\)) for all \(i \in \{1, \ldots, n\}\), and Last(\(\bar{u}\)) denote the conjunction of Last(\(u_i\)) for all \(i \in \{1, \ldots, n\}\). Given \(i \in \{1, \ldots, n\}\), let \(\text{Succ}_i(\bar{u}, \bar{v})\) denote the formula

\[
   u_1 = v_1 \land \cdots \land u_{i-1} = v_{i-1} \land \text{Succ}(u_i, v_i) \\
   \land \text{Last}(u_{i+1}) \land \text{First}(v_{i+1}) \land \cdots \land \text{Last}(u_n) \land \text{First}(v_n).
\]

(80)

Now, by employing the well-known saturation technique (see, e.g., the proof of Theorem 6.3 in [Eiter et al. 1997]), we define the desired logic program II as follows:

\[
   \text{X}(\bar{z}) \lor \tilde{X}(\bar{z}) \quad (X \in \sigma) \quad (81)
\]

\[
   \text{Last}(\bar{x}) \land D(\bar{x}) \rightarrow \text{X}(\bar{z}) \quad (X \in \sigma) \quad (82)
\]

\[
   \text{Last}(\bar{x}) \land D(\bar{x}) \rightarrow \tilde{X}(\bar{z}) \quad (X \in \sigma) \quad (83)
\]

\[
   \text{First}(\bar{x}) \land \tilde{\vartheta}_i(\bar{x}, \bar{y}) \rightarrow D(\bar{x}) \quad (1 \leq i \leq m) \quad (84)
\]

\[
   \text{Succ}_j(\bar{v}, \bar{x}) \land D(\bar{v}) \land \tilde{\vartheta}_i(\bar{x}, \bar{y}) \rightarrow D(\bar{x}) \quad (1 \leq i \leq m, 1 \leq j \leq n) \quad (85)
\]

\[
   \text{Last}(\bar{x}) \land \neg D(\bar{x}) \rightarrow \bot \quad (86)
\]

where \(D\) is an \(n\)-ary fresh predicate; for each predicate \(X \in \sigma\) of arity \(n \geq 0\), \(\tilde{X}\) is a fresh predicate of arity \(n\); and \(\tilde{\vartheta}_i\) is the formula obtained from \(\vartheta_i\) by substituting \(\tilde{X}\) for the expression \(\neg X\) whenever \(X \in \sigma\). Let \(\pi\) denote the predicate set

\[
   \sigma \cup \{D\} \cup \{\tilde{X} : X \in \sigma\}. \quad (87)
\]

It is easy to see that \(\exists\pi\text{SM}(\Pi)\) belongs to \(\text{DLP}^n\). Next we show that \(\exists\pi\text{SM}(\Pi) \equiv_{\text{SUC}} \forall\sigma\varphi\) by employing a similar idea used in the proof of Theorem 6.3 in [Eiter et al. 1997].

Let \(A\) be a successor structure of \(v(\forall \varphi) \cup u_s\) that satisfies \(\forall\sigma\varphi\). Our task now is to prove that \(A\) is a model of \(\exists\pi\text{SM}(\Pi)\). Let \(B\) be an \(\nu(\Pi)\)-expansion of \(A\) in which each of the predicates among \(D, X, \tilde{X} (X \in \sigma)\) is interpreted as the full relation on \(A\) of a proper arity. To obtain the desired conclusion, it is sufficient to show that \(B\) is a stable model of \(\Pi\). It is clear that \(B\) is a model of \(\Pi\). Towards a contradiction, let us assume that \(B\) is not a stable model of \(\Pi\), which implies that there is some assignment \(\beta\) in \(B\) such that

\[
   B \models (\pi^* < \pi)[\beta] \quad & \quad B \models \widehat{\Pi}^*[\beta], \quad (88)
\]

where both notations \(\pi^*\) and \(\widehat{\Pi}^*\) are defined in the same way as those in Section 2. It is not difficult to see that there is some tuple \(\tilde{a} \in A^n\) such that \(\tilde{a} \not\in \beta(D^*)\). Let \(\tilde{b}\) be an arbitrary tuple on \(A\) of length \(|\bar{y}|\). According to rules (84)–(85) it must be true that

\[
   B \models (\neg \tilde{\vartheta}_1(\tilde{a}, \tilde{b}) \land \cdots \land \neg \tilde{\vartheta}_m(\tilde{a}, \tilde{b}))[\beta]. \quad (89)
\]

Let \(\alpha\) be an assignment in \(A\) such that \(\alpha(X) = \beta(X^*)\) for all \(X \in \sigma\). We can infer that

\[
   A \models (\neg \vartheta_1(\bar{a}, \bar{b}) \land \cdots \land \neg \vartheta_m(\bar{a}, \bar{b}))[\alpha], \quad (90)
\]

which is impossible since we have that \(A \models \forall\sigma\varphi\), a contradiction as desired.

For the converse, let us assume that \(A\) is a model of \(\exists\pi\text{SM}(\Pi)\). Towards a contradiction, we also assume that \(A\) is not a model of \(\forall\sigma\varphi\). By the former, we know that there is a stable model \(B\) of \(\Pi\) that is an \(\nu(\Pi)\)-expansion of \(A\). By the definition of \(\Pi\), it is easy to see that \(B\) interprets each predicate among \(D, X, \tilde{X} (X \in \sigma)\) as the full relation on \(A\) of a proper arity. By the assumption that \(A \models \neg \forall\sigma\varphi\), there is an assignment \(\alpha\) in \(A\) such that

\[
   A \models \neg \forall \exists \exists \vartheta(\vartheta_1(\bar{x}, \bar{y}) \lor \cdots \lor \vartheta_m(\bar{x}, \bar{y}))[\alpha]. \quad (91)
\]

20
With this conclusion, we know that there exists a tuple $\vec{a} \in A^n$ such that for all tuples $\vec{b}$ on $A$ of length $|\vec{y}|$ and all integers $i \in \{1, \ldots, m\}$ we have that $A \models \neg \vartheta_i(\vec{a}, \vec{b})[\alpha]$. Let $\beta$ be an assignment in $B$ such that $\beta(D^*) = A^n \setminus \{\vec{a}\}$ and that

$$\beta(X) = \alpha(X) \quad \& \quad \beta(\hat{X}) = A^k \setminus \alpha(X)$$

(92)

for all predicates $X \in \sigma$ of arity $k$. Then, it is a routine task to check that

$$B \models (\pi^* < \pi)[\beta] \quad \& \quad B \models \hat{\Pi}^*[\beta].$$

(93)

From these, we conclude that $B$ is not a model of $SM(\Pi)$, a contradiction as desired. $\square$

Next, we show that this result can be generalized to finite structures. To do this, we need a logic program to define the class of successor structures. Now we define it as follows.

Let $\Pi_0$ denote the normal program that consists of the following rules.

$$\neg \text{Less}(x, y) \rightarrow \text{Less}(x, y)$$

(94)

$$\neg \text{Less}(x, y) \rightarrow \text{Less}(x, y)$$

(95)

$$\text{Less}(x, y) \land \text{Less}(y, z) \rightarrow \text{Less}(x, z)$$

(96)

$$\text{Less}(x, y) \land \text{Less}(y, x) \rightarrow \bot$$

(97)

$$\neg \text{Less}(x, y) \land \neg \text{Less}(y, x) \land \neg x = y \rightarrow \bot$$

(98)

$$\text{Less}(x, y) \rightarrow \text{First}(y)$$

(99)

$$\neg \text{First}(x) \rightarrow \text{First}(x)$$

(100)

$$\text{Less}(x, y) \rightarrow \text{Last}(x)$$

(101)

$$\neg \text{Last}(x) \rightarrow \text{Last}(x)$$

(102)

$$\text{Less}(x, y) \land \text{Less}(y, z) \rightarrow \text{Succ}(x, z)$$

(103)

$$\neg \text{Succ}(x, y) \land \text{Less}(x, y) \rightarrow \text{Succ}(x, y)$$

(104)

Informally, rules (94)–(95) are choice rules to guess a binary relation $\text{Less}$; rules (96), (97) and (98) restrict $\text{Less}$ to be transitive, antisymmetric, and total, respectively, so that $\text{Less}$ is a strict linear order; rules (99)–(100) and rules (101)–(102) then assert that $\text{First}$ and $\text{Last}$ consist of the least and the last elements, respectively, if they indeed exist; and the last two rules state that $\text{Succ}$ defines the relation of immediate successors w.r.t. $\text{Less}$.

The following lemma, which can be proved by a routine check, asserts that the program $\Pi_0$ exactly defines the class of successor structures as we expect.

**Lemma 5.** Let $\sigma \supseteq v_\delta$ be a vocabulary and $A$ be a $\sigma$-structure. Then $A$ is a successor structure iff both $A$ is finite and $A$ is a model of $\exists v\exists SM(\Pi_0)$, where $\tau = v(\Pi_0) \setminus v_\delta$.

Now we can then prove the following result:

**Proposition 7.** $\Sigma_{1,2,n}^1[\forall^n \exists^*] \leq_{\text{FIN}} \text{DLP}_n$ for all $n > 1$.

**Proof.** Let $n > 1$ be an integer, and $\varphi$ be a sentence in $\Sigma_{1,2,n}^1[\forall^n \exists^*]$. Let $\Pi_0$ be the disjunctive program constructed in the proof of Lemma 4 which expresses $\varphi$, and let $\sigma$ be the set of predicates that appear in $\Pi_0$ but neither in $v_\delta$ nor in $v(\varphi)$. Let $\Pi = \Pi_0 \cup \Pi_0$ and let $\tau$ be the set of predicates that appear in $\Pi_0$ but not in $v_\delta$. Next, our task is to show that $\varphi$ is equivalent to $\exists v\exists SM(\Pi)$ over finite structures. By definition and Proposition 2, it suffices to show that $\varphi \equiv_{\text{FIN}} \psi$, where $\psi$ denotes the following formula:

$$\exists v_\delta(\exists v SM(\Pi_0) \land \exists v SM(\Pi_0))).$$

(105)
Let $\nu$ denote the union of $\nu(\varphi)$ and $\nu_0$. Now we prove the new statement as follows.

$\Rightarrow$: Let $A$ be a finite model of $\varphi$. Clearly, there exists an $\nu$-expansion $B$ of $A$ such that $B$ is a successor structure. By Lemma 5, $B$ should be a model of $\exists \sigma SM(\Pi_0)$, and by the proof of Lemma 4, $B$ is also a model of $\exists \sigma SM(\Pi_0)$. Hence, $A$ is a model of $\psi$.

$\Leftarrow$: Let $A$ be a finite model of $\psi$. Then there is an $\nu$-expansion $B$ of $A$ such that $B$ satisfies both $\exists \sigma SM(\Pi_0)$ and $\exists \sigma SM(\Pi_0)$. By Lemma 5, $B$ is a successor structure, and then by Lemma 4, $B$ must be a model of $\varphi$. This means that $A$ is a model of $\varphi$. $\square$

Fix $\nu_n$ as the vocabulary $\{P_n\}$, where $P_n$ is an $n$-ary predicate constant. Let $Parity^n$ denote the class of finite $\nu_n$-structures in each of which $P_n$ is interpreted as a set consisting of an even number of $n$-tuples. The following result was proved by Ajtai.

**Theorem 3** (Implicit by Theorem 2.1 in [Ajtai 1983]). Let $n$ be a positive integer. Then $Parity^n$ is not definable in $\Sigma_{1,n-1}$ over finite structures.

With all of the above results, we are now in the position to establish a weaker separation:

**Theorem 4.** $DLP_n \preceq_{FIN} NLP^f_{2n-1}$ for all $n > 1$.

**Proof.** Fix $n > 1$. Now, let us show that $Parity^{2n}$ is definable in $DLP_n$ over finite structures. We first show that $Parity^{2n}$ is definable in $\Sigma_{2,n}^{v^n \exists^*}$ over successor structures. Let $First$, $Last$ and $Succ_i$ be defined the same on term tuples as those in the proof of Lemma 4, and let $Succ(\bar{s}, \bar{t})$ denote the disjunction of $Succ_i(\bar{s}, \bar{t})$ for all $i \in \{1, \ldots, n\}$ if $\bar{s}$ and $\bar{t}$ are $n$-tuples. Let $X, Y$ be predicate variables of arity $n$, and $\varphi_1$ be the formula

$$\forall \exists(First(\bar{z}) \rightarrow (Y(\bar{z}) \leftrightarrow P_{2n}(\bar{x}, \bar{z}))) \land \forall \exists(\exists Succ(\bar{y}, \bar{z}) \rightarrow (P_{2n}(\bar{x}, \bar{z}) \leftrightarrow Y(\bar{y}) \oplus Y(\bar{z}))),$$

where $\psi \oplus \chi$ denotes the formula $(\psi \leftrightarrow \neg \chi)$ if $\psi$ and $\chi$ are formulas. Intuitively, $\varphi_1$ asserts that $X(\bar{a})$ is true iff the cardinality of $\{\bar{b} \mid (\bar{a}, \bar{b}) \in P_{2n}\}$ is odd. Let $\varphi_2$ denote the formula

$$\forall \exists(First(\bar{z}) \rightarrow (X(\bar{z}) \leftrightarrow Y(\bar{z}))) \land \forall \exists(\exists Succ(\bar{y}, \bar{z}) \rightarrow (X(\bar{y}) \leftrightarrow Y(\bar{y}) \oplus Y(\bar{z}))),$$

Intuitively, $\varphi_2$ denotes $X$ consists of an even number of $n$-tuples on the intended domain. Now, let $\varphi = \exists X \forall Y \forall \exists(\varphi_1 \land \varphi_2)$. Obviously, $\varphi$ can be rewritten as an equivalent sentence in $\Sigma_{2,n}^{v^n \exists^*}$. By a routine check, it is not hard to see that, for every successor structure $A$ of $\nu(\varphi)$, the restriction of $A$ to $\nu_{2n}$ belongs to $Parity^{2n}$ iff $A$ is a model of $\varphi$.

According to the proof of Lemma 4, there exist a disjunctive program $\Pi_0$ and a finite set $\tau$ of predicates of arities $\leq n$ such that $\exists \tau SM(\Pi_0)$ is equivalent to $\varphi$ over successor structures and no predicate in $\nu_0$ is intensional w.r.t. $\Pi_0$. Let $\Pi$ be the union of $\Pi_0$ and $\Pi_0$. Let $\sigma$ be the set of predicates in $\nu(\Pi) \setminus \nu_{2n}$. It is easy to show that, over finite structures, $Parity^{2n}$ is defined by $\exists \sigma SM(\Pi)$, so it is definable in $DLP_n$.

Next, we show that $Parity^{2n}$ is not definable in $NLP^f_{2n-1}$ over finite structures, which yields the desired theorem immediately. By Proposition 6, it suffices to prove that $Parity^{2n}$ is not definable in $\Sigma_{1,2n-1}$ over finite structures. Towards a contradiction, assume it is not true. By a proof similar to Theorem 3.9 (a) in [Durand et al. 1998], it is easy to prove:

**Claim.** Let $m \geq 1$ and suppose the property $Parity^m$ is definable in $\Sigma_{1,m-1}^{f}$ over finite structures. Then the property $Parity^{2m}$ is definable in $\Sigma_{1,2m-2}^{f}$ over finite structures.
With the above claim, we can now conclude that Parity$_{4n}$ is definable in $\Sigma_{4n-2}^{1F}$ over finite structures. Since every function variable of arity $k \geq 0$ can always be simulated by a predicate variable of arity $k + 1$, Parity$_{4n}$ should be definable in $\Sigma_{4n-1}^{1}$ over finite structures, which contradicts with Theorem 3. This completes the proof. \qed

5. ARBITRARY STRUCTURES

Based on the results presented in the previous two sections, we can then compare the expressiveness of disjunctive programs and normal programs over arbitrary structures.

**Theorem 5.** DLP $\simeq$ NLP iff DLP $\simeq_{\text{FIN}}$ NLP.

**Proof.** The direction “$\Longrightarrow$” is trivial. Let us consider the converse. Assume that DLP $\simeq_{\text{FIN}}$ NLP, and let $\Pi$ be a disjunctive program. Then there exists a normal program $\Pi^*$ such that $\text{SM}(\Pi) \equiv_{\text{FIN}} \exists \text{SM}(\Pi^*)$, where $\sigma$ is the set of predicates that occur in $\Pi^*$ but not in $\Pi$. By Theorem 2, there is a normal program $\Pi^{\circ}$ such that $\text{SM}(\Pi) \equiv_{\text{FIN}} \exists \text{SM}(\Pi^{\circ})$.

Without loss of generality, we assume that $\sigma \cap \tau = \emptyset$. To show DLP $\simeq$ NLP, our idea is to design a normal program testing whether the intended structure is finite. We let $\Pi^*$ work if it is true, and let $\Pi^{\circ}$ work otherwise. To do this, we introduce a new predicate $\text{Finite}$ of arity 0, and let $\Pi_f$ be the union of $\Pi_k$ (see Section 4) and the set of rules as follows:

$$\text{First}(x) \rightarrow \text{Num}(x),$$

$$\text{Num}(x) \land \text{Succ}(x, y) \rightarrow \text{Num}(y),$$

$$\text{Num}(x) \land \text{Last}(x) \rightarrow \text{Finite}.$$  \hfill (108, 109, 110)

Let $\pi = \nu(\Pi_f) \setminus \{\text{Finite}\}$. Now let us present a property as follows.

**Claim.** Suppose $A \models \exists \pi \text{SM}(\Pi_f)$. Then $A$ is finite iff $A \models \text{Finite}$.

The direction “$\Longrightarrow$” of this claim follows from Lemma 5. So, it remains to show the converse. Suppose $A$ satisfies $\text{Finite}$. Let $v_0$ denote the union of $v(\Pi_f)$ and the vocabulary of $A$. Then, there exists an $v_0$-expansion $B$ of $A$ such that $B$ is a stable model of $\Pi_f$. Hence, $\text{Less}^B$ should be a strict linear order on $A$; the element in $\text{First}^B$ (respectively, $\text{Last}^B$), if it exists, should be the least (respectively, largest) element in $A$ w.r.t. $\text{Less}^B$; and $\text{Succ}^B$ should be the relation defining the direct successors w.r.t. $\text{Less}^B$. As $\text{Finite}$ is true in $A$, there must exist an integer $n \geq 0$ and $n$ elements $a_1, \ldots, a_n$ from $A$ such that $\text{First}(a_1), \text{Last}(a_n)$ and each $\text{Succ}(a_i, a_{i+1})$ are true in $B$. We assert that, for each $a \in A$ there exists some index $i \in \{1, \ldots, n\}$ such that $a = a_i$. If not, let $b$ be one of such elements. As $\text{Less}^B$ is a strict linear order, there exists $j \in \{1, \ldots, n-1\}$ such that both $\text{Less}(a_j, b)$ and $\text{Less}(b, a_{j+1})$ are true in $B$. But this is impossible since $\text{Succ}(a_j, a_{j+1})$ is true in $B$. So, we have $A = \{a_1, \ldots, a_n\}$. This implies that $A$ is finite as desired.

Next, let us construct the desired program. Let $\Pi_0^\circ$ (respectively, $\Pi_0^\dagger$) denote the program obtained from $\Pi^\circ$ (respectively, $\Pi^\dagger$) by adding $\text{Finite}$ (respectively, $\neg \text{Finite}$) to the body of each rule as a conjunct. Let $\Pi^\dagger = \Pi_0^\dagger \cup \Pi_0^\circ \cup \Pi_f$. Let $\nu = \nu(\Pi^\dagger) \setminus \nu(\Pi)$. Now, we show that $\exists \nu \text{SM}(\Pi^\dagger)$ is equivalent to $\text{SM}(\Pi)$ over arbitrary structures. By definition and Proposition 2, it suffices to show that $\text{SM}(\Pi)$ is equivalent to the formula

$$\exists \text{Finite} \equiv \exists \nu \text{SM}(\Pi_0^\circ) \land \exists \nu \text{SM}(\Pi_0^\circ) \land \exists \nu \text{SM}(\Pi_f)).$$  \hfill (111)

Let $A$ be a structure of $\nu(\Pi)$. As a strict partial order always exists on domain $A$, we can construct an $\nu(\Pi) \cup \nu(\Pi_f)$-expansion $B$ of $A$ such that $B$ is a stable model of $\Pi_f$. 23
By the above claim, $B \models Finite$ if, and only if, $A$ is finite. Let us first assume that $A$ is finite. By definition, it is clear that $\exists \sigma SM(\Pi^1_0)$ is satisfied by $B$ if, and only if, $\exists \sigma SM(\Pi^1_0)$ is satisfied by $A$, and $\exists \sigma SM(\Pi^1_0)$ is always true in $B$. This means that $\exists \sigma SM(\Pi^1_0)$ is equivalent to $SM(\Pi)$ over finite structures. By a symmetrical argument, we can show that the equivalence also holds over infinite structures. This then completes the proof. □

**Remark 7.** In classical logic, it is well-known that separating languages over arbitrary structures is usually easier than that over finite structures [Ebbinghaus and Flum 1999]. In logic programming, it also seems that arbitrary structures are better-behaved than finite structures. For example, there are some preservation theorems that work on arbitrary structures, but not on finite structures [Ajtai and Gurevich 1994]. Therefore, an interesting question then arises as whether it is possible to develop techniques on arbitrary structures for some stronger separations of DLP from NLP.

**Corollary 3.** DLP $\simeq$ NLP iff NP $\equiv$ coNP.

Next, we give a characterization for disjunctive programs.

**Proposition 8.** DLP $\simeq \Sigma^1_1[\forall \exists^*].$

**Proof.** The direction “$\leq$” trivially follows from the second-order definition of stable model semantics. So, we only need to show the converse. To do this, it suffices to prove that, for every first-order sentence $\varphi$ that is of the following form:

$$\forall \sigma \forall \exists \exists \vec{y} (\vartheta_1(\vec{x}, \vec{y}) \lor \cdots \lor \vartheta_m(\vec{x}, \vec{y})), \quad (112)$$

there are a disjunctive program $\Pi$ and a set $\tau$ of auxiliary predicates such that $\varphi$ is equivalent to $\exists \tau SM(\Pi)$, where $\sigma$ is a finite set of predicates; $\vec{x}$ and $\vec{y}$ are two finite tuples of individual variables; and each $\vartheta_i$ is a conjunction of atoms or negated atoms.

Let $n$ be the length of $\vec{x}$. Again by employing the saturation technique (see, e.g., the proof of Theorem 6.3 in [Eiter et al. 1997]), we can construct a logic program $\Pi$ as follows:

$$T_X(\vec{x}, \vec{z}) \lor F_X(\vec{x}, \vec{z}) \quad (X \in \sigma) \quad (113)$$

$$D(\vec{x}) \rightarrow F_X(\vec{x}, \vec{z}) \quad (X \in \sigma) \quad (114)$$

$$D(\vec{x}) \rightarrow T_X(\vec{x}, \vec{z}) \quad (X \in \sigma) \quad (115)$$

$$\vartheta_i(\vec{x}, \vec{y}) \rightarrow D(\vec{x}) \quad (1 \leq i \leq m) \quad (116)$$

$$\neg D(\vec{x}) \rightarrow \bot \quad (117)$$

where, for each $X \in \sigma$, $T_X$ and $F_X$ are two distinct fresh predicates of arity $(n + k)$ if $k$ denotes the arity of $X$; each $\vartheta_i$ is the formula obtained from $\psi_i$ by substituting $F_X(\vec{x}, \vec{t})$ for $\neg X(\vec{t})$ and followed by substituting $T_X(\vec{x}, \vec{t})$ for $X(\vec{t})$ whenever $X \in \sigma$ and $\vec{t}$ is a tuple of terms of a proper length; and $D$ is a fresh predicate of arity $n$. Let $\tau$ denote the set

$$\sigma \cup \{D\} \cup \bigcup_{X \in \sigma} \{T_X, F_X\}. \quad (118)$$

Clearly, $\exists \tau SM(\Pi)$ belongs to DLP. It remains to show that $\exists \tau SM(\Pi)$ is equivalent to $\varphi$.

Let $A$ be a model of $\varphi$, and take $\sigma$ as an arbitrary assignment in $A$. Now our task is to show that $A$ is a model of $\exists \tau SM(\Pi)$. Let $B$ be an $\psi(\Pi)$-expansion of $A$ that interprets each predicate among $D, T_X, F_X (X \in \sigma)$ as the full relation of a proper arity. To obtain the desired conclusion, it suffices to prove that $B$ is a stable model of $\Pi$. It is easy to check
that $B$ is a model of $\tilde{\Pi}$. Towards a contradiction, let us assume that $B$ is not a stable model of $\Pi$, which implies that there exists an assignment $\beta$ in $B$ such that

$$B \models (\pi^* < \pi)[\beta] \quad \& \quad B \models \tilde{\Pi}^*[\beta],$$

where denotes the set $\tau \setminus \sigma$, and both notations $\pi^*$ and $\tilde{\Pi}^*$ are defined in the same way as those in Section 2. From these conclusions, we know that there is some $n$-tuple on $A$, say $\vec{a}$, such that $\vec{a} \not\in \beta(D)^*$. From the conclusion that $B \models \tilde{\Pi}^*[\beta]$, we can infer that

$$B \models \bigwedge_{i=1}^m \neg\exists \vec{y}\tilde{\vartheta}^*_i(\vec{a},\vec{y})[\beta],$$

where $\tilde{\vartheta}^*_i$ denotes the formula obtained from $\tilde{\vartheta}_j$ by substituting $P^*$ for $P$ whenever

$$P \in \bigcup_{X \in \sigma} \{T_X, F_X\}.$$ 

Let $\alpha$ be an assignment in $A$ such that $\alpha(X) = \beta(T^*_X)$ for all predicates $X \in \sigma$. According to the definition of $\tilde{\vartheta}$, it is not difficult to see that

$$A \models \bigwedge_{i=1}^m \neg\exists \vec{y}\vartheta_i(\vec{x},\vec{y})[\alpha],$$

which implies that $A$ is not a model of $\varphi$, a contradiction as desired.

For the converse, let us assume that $A$ is a model of $\exists \pi \text{SM}(\Pi)$. Towards a contradiction, we also assume that $A$ is not a model of $\varphi$. By the former, we know that there is a stable model $B$ of $\Pi$ that is an $\nu(\Pi)$-expansion of $A$. By the definition of $\Pi$, it is easy to see that $B$ interprets each predicate among $D, T_X, F_X \ (X \in \sigma)$ as the full relation on $A$ of a proper arity. By the assumption that $A \models \neg \varphi$, there is an assignment $\alpha$ in $A$ such that

$$A \models \neg \exists \vec{y}\vartheta_1(\vec{x},\vec{y}) \lor \cdots \lor \vartheta_m(\vec{x},\vec{y})[\alpha].$$

With this, we can infer that there exists an index $i \in \{1, \ldots, m\}$ such that

$$A \models \neg \exists \vec{y}\vartheta_i(\vec{x},\vec{y})[\alpha].$$

Let $\vec{a} = \alpha(\vec{x})$, and let $\beta$ be an assignment in $B$ such that $\beta(D^*) = A^n \setminus \{\vec{a}\}$ and that

$$\beta(T^*_X) = \{(\vec{a}, \vec{b}) \mid \vec{b} \in \alpha(X)\} \cup \{\vec{a}_0, \vec{b} \mid \vec{a}_0 \in A^n \setminus \{\vec{a}\} \land \vec{b} \in A^k\},$$

$$\beta(F^*_X) = \{(\vec{a}, \vec{b}) \mid \vec{b} \in A^k \setminus \alpha(X)\} \cup \{\vec{a}_0, \vec{b} \mid \vec{a}_0 \in A^n \setminus \{\vec{a}\} \land \vec{b} \in A^k\}$$

for all predicates $X \in \sigma$ of arity $k$. Then, it is a routine task to check that

$$B \models (\pi^* < \pi)[\beta] \quad \& \quad B \models \tilde{\Pi}^*[\beta],$$

where $\pi$ denotes the set of predicates $D$ and $T_X, F_X$ for all $X \in \sigma$, and both notations $\pi^*$ and $\tilde{\Pi}^*$ are defined in the same way as those in Section 2. From this, we then conclude that $B$ is not a model of $\text{SM}(\Pi)$, i.e., a stable model of $\Pi$, a contradiction as desired. □

6. CONCLUSION AND RELATED WORK

We have undertaken a comprehensive study on the expressiveness of logic programs under the general stable model semantics. From the results we proved in this paper and other existing results, now we can draw an almost complete picture for the expressiveness of logic
programs and some related fragments of second-order logic. As shown in Figure 1, the expressiveness hierarchy in each table is related to a structure class, while the syntactical classes in a same block is proved to be of the same expressiveness over the corresponding structure class. The closer a block is to the top, the more expressive the classes in the block are. In addition, a dashed line between two blocks indicates that the corresponding separation is true if, and only if, the complexity class NP is not closed under complement.

![Diagram of expressiveness hierarchies related to logic programs](image)

**Fig. 1.** Expressiveness Hierarchies Related to Logic Programs

Without involving the well-known conjecture in Complexity Theory, we established the intranslatability from disjunctive programs to normal programs over finite structures if the arities of auxiliary predicate and function symbols are bounded in a certain sense. This can be regarded as a strong evidence that disjunctive programs are more expressive than normal programs over finite structures. As a byproduct, we also developed a succinct translation from normal logic programs to first-order sentences. This may be viewed as an alternative to the work of the ordered completion for translating a normal program into a first-order sentence over finite structures proposed in [Asuncion et al. 2012]. It is also worth noting that both the number of auxiliary symbols and the maximum arity of auxiliary symbols involving in our translation are smaller than those in the ordered completion.

There are several existing works that contribute to Figure 1, which are listed as follows. The translatability from $\Sigma^1_2$ to $\Sigma^1_2[\forall^*\exists^*]$ follows from the well-known Skolem normal form. The translatability from $\Sigma^1_2$ to $\Sigma^1_2[\forall^*\exists^*]$ over finite structures is due to [Leivant 1989]. The separation of $\Sigma^1_2$ from $\Sigma^1_2[\forall^*\exists^*]$ (on both arbitrary and infinite structures) is implicit in [Eiter et al. 1996]. The translatability from DLP to $\Sigma^1_2[\forall^*\exists^*]$ over arbitrary structures (so also over infinite structures and over finite structures) directly follows from the second-order transformation [Ferraris et al. 2011]. From NLP to $\Sigma^1_1$, the intranslatability over finite structures (so also over arbitrary structures) is due to [Asuncion et al. 2012].

Over Herbrand structures, [Schlipf 1995; Eiter and Gottlob 1997] proved that normal programs, disjunctive programs and universal second-order logic are of the same expressiveness under the query equivalence. Their proofs employ an approach from Computability Theory. However, this approach seems difficult to be applied to general infinite structures. In the propositional case, there have been a lot of works on the translatability and expressiveness of logic programs, e.g., [Eiter et al. 2004; Janhunen 2006]. It should be noted that the picture of expressiveness and translatability in there is quite different from that in the first-order case. For example, it is not difficult to show that every boolean function can be defined by a normal program if auxiliary propositional symbols are allowed, and thus developing a translation from propositional disjunctive programs to propositional normal...
programs is always possible if we do not consider the succinctness. There have been also some works (see, e.g., [Eiter et al. 2013]) that focus on translatability between classes of propositional logic programs under the strong equivalence and uniform equivalence. As a future work, it would be interesting to generalize our work to these equivalence.

Translations from logic programs into first-order theories had been also investigated in several earlier works. Chen et al. [2006] proved that, over finite structures, every normal program with variables can be equivalently translated to a possibly infinite first-order theory. This result was later extended to disjunctive programs [Lee and Meng 2011]. In addition, Lee and Meng [2011] identified the intranslatability from logic programs into possibly infinite first-order theories over arbitrary structures, and proposed some sufficient conditions which assure the translatability over arbitrary structures. Instead of using possibly infinite theories, translations in this paper only involve finite theories. The translatability between logic programs and first-order theories was also considered in [Zhang et al. 2011], but first-order theories there are equipped with the general stable model semantics.

It is also worth mentioning some related works that focus on identifying sufficient conditions for the translatability between first-order logic programs and first-order theories. A possibly incomplete list is as follows. Ferraris et al. [2011] showed that every tight logic program is equivalent to a first-order sentence. Chen et al. [2011] proposed a notion called loop-separability that assures the first-order definability over finite structures. Zhang and Zhang [2013b] established some semantic characterizations for the first-order expressibility of negation-free disjunctive programs over arbitrary structures. Ben-Eliyahu and Dechter [1994] proved that every head-cycle-free disjunctive program is equivalent to a normal program. Very recently, Zhou [2015] proposed a semantic notion called choice-boundedness that assures the translatability from disjunctive programs to normal programs. All these results provide another view to understand the expressiveness of logic programs.

REFERENCES


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