Properties of Iterated Multiple Belief Revision

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Abstract. In this paper we investigate the properties of iterated multiple belief revision. We examine several typical assumptions for iterated revision operations with an ontology where an agent assigns ordinals to beliefs, representing strength or firmness of beliefs. A notion of minimal change is introduced to express the idea that if no evidence to show how a belief set should be reordered after it is revised, the changes on the ordering should be minimal. It has been shown that under the assumption of minimal change, the multiple version of Darwiche and Pearl's postulate (C1) holds no matter in what degree new information is accepted. Moreover, under the same assumption, Boutilier's postulate (CB) holds if and only if new information is always accepted in the lowest degree of firmness while Nayak *et al.*'s postulate (CN) holds if and only if new information is always accepted in the highest degree. These results provide an ontological base for analyzing the rationality of postulates of iterated belief revision.
Keywords: iterated belief revision, multiple belief revision, belief revision

1 Introduction

Iterated belief revision has been intensively investigated in the community of belief revision[2][5][9][10][13][17]. The major concern in the research is the reducibility of iterated revisions to single step revisions. Boutilier in [2] proposed an assumption to capture the relationship, named (CB) by Darwiche and Pearl in [4]:

(CB) if $K * A \vdash \neg B$, then $(K * A) * B = K * B$

Darwiche and Pearl in [4] argued that (CB) would be overcommitted and could be weakened and enhanced elsewhere with the following assumptions:

($C1$) If $B \vdash A$, then $(K * A) * B = K * B$.
($C2$) If $B \vdash \neg A$, then $(K * A) * B = K * B$.
($C3$) If $A \in K * B$, then $A \in (K * A) * B$.
($C4$) If $\neg A \notin K * B$, then $\neg A \notin (K * A) * B$.

It was shown by Freud and Lehmann (see [10]), however, that these postulates do not go well with AGM framework: $(C2)$ *is inconsistent with AGM postulates*. Nevertheless, Darwiche and Pearl argued in [5] that the original AGM postulates should be weakened in order to accommodate the additional postulates for iterated revision. More other iterated belief revision frameworks have been also proposed in the last few years, reflecting different philosophy of iterated belief revision[9][10][14]. No matter how much divergency these frameworks have made it has been generally accepted that belief change should not be considered as a purely set-theoretical change of belief sets but an evolution of epistemic states, which encapsulate beliefs with the information of firmness of beliefs. The change of epistemic states involves not only the change on belief set but also the change on orderings over the belief set. Whenever a belief set is revised by new information, the ordering of the belief set should also change to accommodate the new information. The further revision will be based on the new ordering. Therefore the posterior revision operation is normally not identical with the prior operation. This idea has been explicitly expressed by Nayak *et al.* in [14] by using a subscription to differentiate two steps revision. For instance, $(C1)$ and $(C2)$ can be restated in the following form:

$(C1')$ If $B \vdash A$, then $(K * A) *_A B = K * B$.
$(C2')$ If $B \vdash \neg A$, then $(K * A) *_A B = K * B$.[1]

We remark that this change is significant since the postulates will no long lay restrictions on single step revisions but to specify the relationship between the first step revision and the second step revision. In fact, it has been shown in [14] that $(C1')$ and $(C2')$ are consistent with AGM postulates. Therefore we can use these additional postulates to extend the AGM framework without revising the AGM postulates. Moreover, Nayak *et al.* in [14] even showed that $(C1')$ can be strengthened further by the following postulate without loss of consistency:

(CN) If $A \wedge B \not\vdash \bot$, then $(K * A) *_A B = K * (A \wedge B)$.

Most of the research in iterated revision based on single belief revision, that is, the new information is represented by a single sentence. It is even more interesting if we consider the problem in the setting of multiple belief revision. By using the notation introduced in [22] and Nayak *et al.*'s notation, all the above postulates can be restated in the following form:

($\otimes C1$) $(K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2) = K \otimes (F_1 \cup F_2)$.
($\otimes C2$) If $F_1 \cup F_2$ is inconsistent, then $(K \otimes F_1) \otimes_{F_2} F_2 = K \otimes F_2$.
($\otimes C3$) If $F_1 \subseteq K \otimes F_2$, then $F_1 \subseteq (K \otimes F_1) \otimes_{F_1} F_2$.
($\otimes C4$) If $F_1 \cup (K \otimes F_2)$ is consistent, then $F_1 \cup ((K \otimes F_1) \otimes_{F_1} F_2)$ is consistent.
($\otimes CB$) If $F_2 \cup (K \otimes F_1)$ is inconsistent, then $(K \otimes F_1) \otimes_{F_1} F_2 = K \otimes F_2$.
($\otimes CN$) If $F_1 \cup F_2$ is consistent, $(K \otimes F_1) \otimes_{F_1} F_2 = K \otimes (F_1 \cup F_2)$.

Surprisingly, the extension does not increase the complexity of these postulates. Contrarily, they become more readable. For instance, $(\otimes C1)$ expresses the assumption that *if the second revision confirms all the information contained in the first one, the first revision is superfluous*[10].

One question is to be answered that whether these postulates are consistent with the multiple version of AGM postulates, especially with the Limit Postulate introduced by Zhang *et. al.* in [19]. Another question, which is more important, that what are the

[1] In [14] an extra condition $\not\vdash \neg A$ is added to avoid the treatment of inconsistent belief set.

underlying ontology to use these postulates. As Friedman and Halpern pointed out in [6] that there has been too much attention paid to postulates and not enough to the underlying ontology. We need to make it clear that *in what situation an agent revises her epistemic states in the way that we specified by these postulates*.

In this paper we exploit an ontology where an agent assigns an ordinal to each of her belief, representing the strength or firmness of the belief[2]. We introduce a notion of minimal change of belief degrees, which expresses the idea that if no evidence to show that how a belief set should be ordered after it is revised, the change of the ordering on the belief set should be minimal. We show that under the assumption of minimal change of ordering, $(\otimes C1)$ holds no matter in what degree the new information is believed. However, the postulates $(\otimes CN)$ and $(\otimes CB)$ heavily depend on how the new information is accepted. We show that $(\otimes CN)$ holds if and only if the new information is accepted and kept in the highest degree of firmness whereas $(\otimes CB)$ holds if and only if the new information is accepted in the lowest degree comparing with the old beliefs. These results provide an ontological base for the analysis of the rationality on the postulates for iterated belief revision.

Throughout this paper, we consider a propositional language $\mathcal{L}$ as the object language. We denote individual sentences in $\mathcal{L}$ by A, B, or C, and denote sets of sentences by F,F_1, F_2 etc. If F is an infinite set of sentences, $\bar{F}$ denotes a finite subset of F. If F is a finite set, $\wedge F$ means the conjunction of its elements. We shall assume that the underlying logic includes the classical first-order logic with the standard interpretation. The notation $\vdash$ means the classical first-order derivability and Cn the corresponding closure operator, i.e.,

$$A \in Cn(\Gamma) \text{ if and only if } \Gamma \vdash A$$

A set K of sentences is a belief set if $K = Cn(K)$.

2 Preliminaries in Multiple Belief Revision

Firstly, let's review the basic concepts in mutual belief revision. Zhang and Foo in [22] introduced a version of multiple belief revision, called *set revision*, which allows to revise a belief set by any set of sentence or another belief set. Nine postulates were proposed. Among them the first eight postulates are the direct generalization of the associated AGM ones(also see [12][13]). The last one, called *Limit Postulate*, deals with the compactness of infinite belief revision [22], which says that a revision by an infinite belief set can be approached by the revisions of the belief set with its finite subsets.

Formally, let $\mathcal{K}$ be the set of all belief sets. A function $\otimes$ is called a *set revision function* if it satisfies the following postulates:

$(\otimes 1)$ $K \otimes F = Cn(K \otimes F)$.

[2] In fact, the ordinal will be interpreted as the strength of disbelief due to technical reason.

$(\otimes 2)$ $F \subseteq K \otimes F$.
$(\otimes 3)$ $K \otimes F \subseteq K + F$.
$(\otimes 4)$ If $F \cup K$ is consistent, then $K + F \subseteq K \otimes F$.
$(\otimes 5)$ $K \otimes F$ is inconsistent if and only if F is inconsistent.
$(\otimes 6)$ If $Cn(F_1) = Cn(F_2)$, then $K \otimes F_1 = K \otimes F_2$.
$(\otimes 7)$ $K \otimes (F_1 \cup F_2) \subseteq (K \otimes F_1) + F_2$.
$(\otimes 8)$ If $F_2 \cup (K \otimes F_1)$ is consistent, then $(K \otimes F_1) + F_2 \subseteq K \otimes (F_1 \cup F_2)$.
$(\otimes LP)$ $K \otimes F = \bigcup_{\bar{F} \subseteq F} \bigcap_{\substack{\bar{F}' \subseteq_f Cn(F) \\ \bar{F} \subseteq \bar{F}'}} K \otimes \bar{F}'$.

where K is a belief set. F, F_1, F_2 are sets of sentences. $\bar{F} \subseteq_f F$ means $\bar{F}$ is a finite subset of F.

The model of set revision is based on the following variant of epistemic entrenchment.

Definition 1. [18] Let K be a belief set, $\mathcal{P}$ a partition of K, and $<$ a total order over $\mathcal{P}$. The triple $K = (K, \mathcal{P}, <)$ is called *a total-ordered partition(TOP)* of K. For any $P \in \mathcal{P}$ and $A \in P$, P is called the *rank* of A, denoted by $r(A)$.

A NOP is called a *nicely-ordered partition* (NOP) if it satisfies the following *Logical Constraint*:

If $A_1, \cdots, A_n \vdash B$, then $r(B) \leq \max\{r(A_1), \cdots, r(A_n)\}$.

It is easy to see that a TOP is nothing but a total pre-order on belief set. The ordering $<$ in an NOP is essentially the reverse order of epistemic entrenchment(EE). Logical Constraint here is the combination of (EE2) and (EE3)[7]. (EE5) does not satisfied by NOP. Therefore NOP is weaker version of EE (see more detail about the relationship between NOP and EE in [22]).

With the notion of NOP, a set revision operator can be constructed as follows:

Definition 2. [19] Let $\Sigma = (K, \mathcal{P}, <)$ be a NOP of a belief set K. Define a revision function $\otimes$ as follows: for any set F of sentences,

i). If $F \cup K$ is consistent, then $K \otimes F = K + F$; otherwise,

ii). $B \in K \otimes F$ if and only if there exists $A \in K$ such that $F \vdash \neg A$ and

$$\forall C \in K((A \vdash C \ \&\ F \vdash \neg C) \Rightarrow (r(C \vee B) < r(C) \text{ or } \vdash C \vee B)) \quad (1)$$

$\otimes$ is called an *NOP-based revision.*

It has been shown in [22] that *a set revision operator satisfies the nine postulates if and only if it is a NOP-based revision*[3].

3 Iterated Multiple Belief Revision

Now we consider iterated operations in the setting of multiple belief revision. Firstly, let's consider the relationship between the additional postulates.

[3] The original representation theorem was given for contraction operator. The representation result for revision operator can be easily obtained by using Levi Identity.

3.1 Relationship of iterated revision postulates

The following lemma shows the relationship between the postulates for iterated revision and AGM postulates.

Lemma 1. *Let $\otimes$ be a revision function that satisfies ($\otimes 1$)-($\otimes 6$). Then*

i). ($\otimes C1$) implies ($\otimes 7$) and ($\otimes 8$).
ii). ($\otimes CB$) implies ($\otimes C1$) and ($\otimes C2$).
iii). ($\otimes CN$) implies ($\otimes C1$), ($\otimes C3$) and ($\otimes C4$).

Due to the relationship we will concentrate on the postulate $(\otimes C1)$, $(\otimes CB)$ and $(\otimes CN)$ only in the sequent of the paper. We will examine the conditions when the postulates hold. First, let's find an ontology where these conditions can be stated.

3.2 Perfectly-ordered partition

In [20], Zhang *et al.* introduced a special kind of nice-ordered partition in which the partition is well-ordered.

Definition 3. [20] A nicely-ordered partition $\Sigma = (K, \mathcal{P}, <)$ is called a *perfectly-ordered partition* (POP) if $<$ is a well-order on $\mathcal{P}$. An NOP-based revision function is a *POP-based revision* if it is generated by a perfectly-ordered partition.

Mapping a sentence to a natural number or an ordinal is one of the traditional ways to rank beliefs. Applying logical constraint on the mapping to make it to be an epistemic entrenchment ordering has been also exploited in [17][18]. However, the ordering we use here should be differentiated from the ways to map possible worlds to ordinals[16] or to map Spohn's system of spheres to ordinals[15] since the underlying ontologies are significantly different even though they might be interconvertible.

The following lemmas show the logical properties of a POP.

Lemma 2. *Let $\Sigma = (K, \mathcal{P}, <)$ be a POP and η the ordinal type of $<$. Let $P_\alpha = \{A \in K : r(A) = \alpha\}$. Let $P_{\leq\alpha} = \bigcup_{\beta\leq\alpha} P_\beta$. Then for any $\alpha < \eta$, $P_{\leq\alpha}$ is logically closed.*

Conversely, let r be a function mapping a belief set K to an ordinal η. If, for any $\alpha < \eta$, $\{A \in K : r(A) \leq \alpha\}$ is logically closed, then r determines a POP over K.

Lemma 3. *Let $\otimes$ be a revision function generated by a POP $\Sigma = (K, \mathcal{P}, <)$. For any set F of sentences, if $F \cup K$ is inconsistent, then $B \in K \otimes F$ if and only if there exists a sentence $A \in K$ such that $F \vdash \neg A$ and*

$$b(A \vee B) < P_{min} \text{ or } \vdash A \vee B$$

where $P_{min} = \min\{r(A) : A \in K \ \& \ F \vdash \neg A\}$.

3.3 Minimal change of belief degrees

The theory of belief revision is dominated by the principle of minimal change. Such minimal change should not mean the change in cardinality of belief sets but mean the change of epistemic states: belief sets and associated degrees of beliefs. Such a distinction is not essential in one shot revision but becomes crucial in iterated revisions. An iterated revision operation should be viewed as a function with three inputs (*belief set, new information, ordering over the belief set*) and two outputs (*new belief set, new ordering*). The change of the ordering during revision partially determines the relationship between one step revision and iterated revision. Therefore different philosophy of minimal change of ordering leads to different postulates for iterated revision[2][13]. The following definition depicts an instantiation of the principle of minimal change that the ordering on beliefs should keep unchanged unless it violates the Logical Constraint.

Definition 4. Let $\otimes$ be a revision function based on a POP $\Sigma = (K, \mathcal{P}, <)$ and η be the order type of $\mathcal{P}$. For any set F of sentences and an ordinal α, define a POP $\Sigma^{F,\alpha} = (K \otimes F, \mathcal{P}^{F,\alpha}, <^{F,\alpha})$ over $K \otimes F$ as follows:

1. For any $\beta < \max\{\eta, \alpha + 1\}$,
   $P_\beta^{F,\alpha} = \begin{cases} P_\beta \cap (K \otimes F), & \text{if } \beta < \alpha; \\ (Cn(P_{\leq\beta} \cup F) \cap K \otimes F) \backslash P_{<\beta}^{F,\alpha}, & \text{otherwise.} \end{cases}$
   where $P_{\leq\beta} = \bigcup\limits_{\gamma\leq\beta} P_\gamma$ and $P_{<\beta}^{F,\alpha} = \bigcup\limits_{\gamma<\beta} P_\gamma^{F,\alpha}$.
2. Let $\mathcal{P}^{F,\alpha} = \{P_\beta^{F,\alpha} : \beta < \max\{\eta, \alpha + 1\}\}$.
3. For any $P_\beta^{F,\alpha}$, $P_\gamma^{F,\alpha} \in \mathcal{P}^{F,\alpha}$, define:

$$P_\beta^{F,\alpha} <^{F,\alpha} P_\gamma^{F,\alpha} \text{ if and only if } \beta < \gamma$$

We call $\Sigma^{F,\alpha}$ the *minimal change of Σ with respect to F and α.*

The intuition behind the definition is that if we accept the new information F with the degree of α, F and its logical consequence with K will be merged to the old partition in the way that the ordering of partition will keep unchange except the belief degree of some $K \cup F$'s logical consequence could be higher because them are enhanced by new information.

The following theorem shows that under the assumption of minimal change of POP, $(\otimes C1)$ holds no matter in what degree the new information is accepted.

Theorem 1. *Let $\otimes$ be a revision function based on a POP $\Sigma = (K, \mathcal{P}, <)$. Let $\otimes_{F_1}$ be the revision function based on a minimal change $\Sigma^{F_1,\alpha}$ of Σ. Then we have*

$(\otimes C1)$ $\quad (K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2) = K \otimes (F_1 \cup F_2)$.

Corollary 1. *($\otimes$1)-($\otimes$6) and ($\otimes LP$) are consistent with ($\otimes C1$).*

4 Two Radical Strategies of Accepting New Information

We have seen that, provided the change of belief degrees is minimal, $(\otimes C1)$ holds no matter where the new information is inserted. However, the strategy to accept new information significantly affects the result of iterated revision. In this section, we consider two radical strategies of accepting new information: extreme cases that new information is accepted in absolute affirmative and in extreme suspicion. The following theorem deals with the first case.

Theorem 2. *Let $\otimes$ be a revision function based on POP $\Sigma = (K, \mathcal{P}, <)$ and $\otimes_{F_1}$ the revision function based on Σ's minimal change $\Sigma^{F_1,\alpha}$. If $\alpha = 0$ and $P_0 \subseteq Cn(F_1)$, then we have*

$(\otimes CN)$ *If $F_1 \cup F_2$ is consistent, $(K \otimes F_1) \otimes_{F_1} F_2 = K \otimes (F_1 \cup F_2)$.*

The theorem shows that $(\otimes CN)$ holds if new information is always accepted in the top level of firmness(no weaker than any old beliefs). Surprisingly, the condition is not only sufficient but also necessary except for some limit cases. It is not hard to verify that $(\otimes CN)$ holds for any α if one of the following conditions is true:

1. $F_1 = Cn(\emptyset)$;
2. $P_0 \cup F_1$ *is inconsistent;*
3. *there is* β *such that* $P_{\leq \beta} = Cn(F_1)$.

By excluding these limit cases we have

Proposition 1. *If $F_1 \neq Cn(\emptyset)$, $P_\alpha \cup F_1$ is consistent and $P_\alpha \neq Cn(F_1)$, then that $(\otimes CN)$ implies $P_{\leq \alpha} \subseteq Cn(F_1)$.*

Therefore, only if the new information is accepted in the level higher than any other old information, $(\otimes CN)$ is satisfied.

Next we consider another extreme strategy of inserting new information: *the new information is accepted in the lowest degree*.

Theorem 3. *Let $\otimes$ be a revision function based on a POP $\Sigma = (K, \mathcal{P}, <)$ with ordinal type η. Let $\otimes_{F_1}$ be the revision function based on the minimal change $\Sigma^{F_1,\alpha}$ of Σ. If $\alpha \geq \eta$, then*

$(\otimes CB)$ *If $F_2 \cup (K \otimes F_1)$ is inconsistent, then $(K \otimes F_1) \otimes_{F_1} F_2 = K \otimes F_2$.*

The necessary condition for $(\otimes CB)$ is presented in the following proposition.

Proposition 2. *If $F_1 \not\subseteq K$, then $(\otimes CB)$ implies $K \otimes F_1 \subseteq P^{F_1,\alpha}_{\leq \alpha}$.*

Therefore to satisfy $(\otimes CB)$, new information should be accepted in the lowest degree unless it has been included in the belief set.

5 Conclusion and Discussion

In this paper we have presented a model-theoretical analysis of several typical postulates for iterated belief revision in the setting of multiple belief revision. The model we use in the paper is one of the typical ways to rank beliefs: assigning beliefs to ordinals. We have shown that Darwiche and Pearl's postulate $(\otimes C1)$ requires the change of orderings to be minimal. It doesn't put any restrictions on the way to rank new information. However, as the strengthening of $(\otimes C1)$, Boutilier's postulate $(\otimes CB)$ implies that new information should be accepted in the lowest degree of belief whereas Nayak *et al.*'s postulate $(\otimes CN)$ requires that new information should be accepted in highest level of firmness. These results provide an ontological base for the analysis of rationality of postulates for iterated belief revision.

As a side-product, we have proved that all the postulates discussed in the paper are consistent with Zhang and Foo's postulates for multiple revision. Therefore the multiple revision framework can be strengthened by choosing some of the postulates. Since the AGM postulates and Darwiche and Pearl's postulate $(\otimes C1)$ have received strong ontological support and are most intuitive. A framework that consists of these postulates would be most applicable. Zhang *et al.* used the framework in the construction of mutual revision functions and negotiation functions[23], which provides another ontology of iterated belief revision.

Proofs of Theorems

Proof of Lemma 1: i). For ($\otimes$7), $K\otimes(F_1\cup F_2) = (K\otimes F_1)\otimes_{F_1}(F_1\cup F_2) \subseteq (K\otimes F_1)+(F_1 \cup F_2) = (K \otimes F_1) + F_2$. To show ($\otimes$8), assume that $F_2 \cup (K \otimes F_1)$ is consistent. Then $(K\otimes F_1)+F_2 = (K\otimes F_1)+(F_1\cup F_2) = (K\otimes F_1)\otimes_{F_1}(F_1\cup F_2) = K\otimes(F_1\cup F_2)$.

ii) and ii) are straightforward. □.

Proof of Lemma 2: For the first part of the lemma, assume $A \in Cn(P_{\leq\alpha})$. Then there exist $A_1, \cdots, A_n \in P_{\leq\alpha}$ such that $A_1, \cdots, A_n \vdash A$. By Logical Constraint, $\max\{r(A_1), \cdots, r(A_n)\} \leq r(A)$. Therefore $A \in P_{\leq\alpha}$.

For the second part of the lemma, let $\mathcal{P} = \{P_\alpha : \alpha < \eta\}$, where $P_\alpha = \{A \in K : r(A) = \alpha\}$. It is easy to verify that $\mathcal{P}$ satisfy the logical constraint. □

Proof of Lemma 3: Suppose that $F \cup K$ is inconsistent. Then $\{r(A) : A \in K$ and $F \vdash \neg A\}$ is nonempty. Since $\mathcal{P}$ is well-ordered, there must be a minimum P_{min} in the set.

Now assume that $B \in K \otimes F$. According to Definition 2, there exists $A_0 \in K$ such that $F \vdash \neg A_0$ and

$$\forall C \in K((A_0 \vdash C \ \&\ F \vdash \neg C) \Rightarrow (r(C \vee B) < r(C) \text{ or } \vdash C \vee B)) \tag{2}$$

Suppose that $r(A_1) = P_{min}$ and $F \vdash \neg A_1$. Let $A = A_0 \vee A_1$. Then $F \vdash \neg A$. By using Equation 2, we obtain $r(A \vee B) < r(A)$ or $\vdash A \vee B$, that is, $r(A \vee B) < P_{min}$ or $\vdash A \vee B$.

Conversely, assume that $A \in K$, $F \vdash \neg A$ and

$$b(A \vee B) < P_{min} \text{ or} \vdash A \vee B$$

If $\vdash A \vee B$, then obviously $B \in K \otimes F$. Otherwise, for any $C \in K$, if $F \vdash \neg C$ and $A \vdash C$, then $r(C) \geq P_{min}$. Thus $r(C \vee B) \leq r(A \vee B) < P_{min} \leq r(C)$. By Definition 1 we conclude that $B \in K \otimes F$. $\Box$

Proof of Theorem 1: In the case that $(F_1 \cup F_2) \cup K$ or $(F_1 \cup F_2) \cup (K \otimes F_1)$ is consistent, the result is straightforward. Thus we assume that both $(F_1 \cup F_2) \cup K$ and $(F_1 \cup F_2) \cup (K \otimes F)$ are inconsistent. Let

$P_{min}^{F_1 \cup F_2} = \min\{r(A) : A \in K \ \& \ F_1 \cup F_2 \vdash \neg A\} = \beta$,

$P_{min\otimes}^{F_1 \cup F_2} = \min\{r^1(A) : A \in K \otimes F_1 \ \& \ F_1 \cup F_2 \vdash \neg A\} = \gamma$,

where r and r^1 is the rank with respect to Σ and $\Sigma^{F_1, \alpha}$, respectively. Then there exists $A_0 \in K$ such that $F_1 \cup F_2 \vdash \neg A_0$ and $r(A_0) = \beta$. Similarly, there exists $A_1 \in K \otimes F_1$ such that $F_1 \cup F_2 \vdash \neg A_1$ and $r^1(A_1) = \gamma$. Let $A = A_0 \vee A_1$. Then $F_1 \cup F_2 \vdash \neg A$, $A \in K$ and $A \in K \otimes F_1$. By the minimality of $P_{min}^{F_1 \cup F_2}$ and $P_{min\otimes}^{F_1 \cup F_2}$, we know that $r(A) = \beta$ and $r^1(A) = \gamma$.

Next we prove $\beta = \gamma$. On one hand, since $A \in K \cap (K \otimes F_1)$, by the construction of minimal change, $r^1(A) \leq r(A)$. Thus $\gamma \leq \beta$. On the other hand, $r^1(A) = \gamma$ implies that $A \in P_{\gamma}^{F_1, \alpha}$. Hence $P_{\leq \gamma} \cup F_1 \vdash A$. It follows that there exists a finite subset $\bar{F}_1$ of F_1 such that $P_{\leq \gamma} \vdash \neg(\wedge \bar{F}_1) \vee A$. Therefore $r(\neg(\wedge \bar{F}_1) \vee A) \leq \gamma$. Since $F_1 \cup F_2 \vdash (\wedge \bar{F}_1) \wedge \neg A$ and $\neg(\wedge \bar{F}_1) \vee A \in K$, by the minimality of β we have $r(\neg(\wedge \bar{F}_1) \vee A) \geq \beta$. Therefore we obtain that $\beta \leq r(\neg(\wedge \bar{F}_1) \vee A) \leq \gamma$, that is, $\beta \leq \gamma$.

Now we prove that $(K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2) = K \otimes (F_1 \cup F_2)$. Assume that $B \in (K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2)$. Then there exists $A_2 \in K \otimes F_1$ such that $F_1 \cup F_2 \vdash \neg A_2$ and

$$r^1(A_2 \vee B) < P_{min\otimes}^{F_1 \cup F_2} \text{ or } \vdash A_2 \vee B.$$

If $\vdash A_2 \vee B$, then $\neg A_2 \vdash B$. Therefore $B \in K \otimes (F_1 \cup F_2)$. If $r^1(A_2 \vee B) < P_{min\otimes}^{F_1 \cup F_2}$, let $r^1(A_2 \vee B) = \delta$. Then $P_{\leq \delta} \cup F_1 \vdash A_2 \vee B$. It follows that there exists a finite subset $\bar{F}_1$ of F_1 such that $P_{\leq \delta} \vdash \neg(\wedge \bar{F}_1) \vee A_2 \vee B$. Let $A_2' = \neg(\wedge \bar{F}_1) \vee A_2 \vee A$, where A was defined above. Therefore $r(A_2' \vee B) \leq \delta < P_{min\otimes}^{F_1 \cup F_2} = P_{min}^{F_1 \cup F_2}$. Since $A_2' \in K$ and $F_1 \cup F_2 \vdash \neg A_2'$, by Lemma 3, we yield that $B \in K \otimes (F_1 \cup F_2)$. We have proved $(K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2) \subseteq K \otimes (F_1 \cup F_2)$.

For the other direction, assume that $B \in K \otimes (F_1 \cup F_2)$. Then there exists $A_3 \in K$ such that $F_1 \cup F_2 \vdash \neg A_3$ and

$$r(A_3 \vee B) < P_{min}^{F_1 \cup F_2} \text{ or } \vdash A_3 \vee B.$$

The case of $\vdash A_3 \vee B$ is obvious. In the case of $r(A_3 \vee B) < P_{min}^{F_1 \cup F_2}$, let $A_3' = A_3 \vee A$, where A was defined above. Then $A_3' \in K \otimes F_1$ and $F_1 \cup F_2 \vdash \neg A_3'$. It follows that

$$r^1(A_3' \vee B) \leq r(A_3' \vee B) \leq r(A_3 \vee B) < P_{min}^{F_1 \cup F_2} = P_{min\otimes}^{F_1 \cup F_2}$$

By Lemma 3, we yield that $B \in (K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2)$. Therefore $K \otimes (F_1 \cup F_2) \subseteq (K \otimes F_1) \otimes_{F_1} (F_1 \cup F_2)$. $\Box$

Proof of Theorem 2: For sufficiency, assume that $(F_1 \cup F_2) \cup K$ is consistent. Then $(K \otimes F_1) \otimes_{F_1} F_2 = (K + F_1) + F_2 = K + (F_1 \cup F_2) = K \otimes (F_1 \cup F_2)$. If $F_2 \cup (K \otimes F_1)$ is consistent, $(K \otimes F_1) \otimes_{F_1} F_2 = (K \otimes F_1) + F_2$. By $(\otimes 7)$ and $(\ominus 8)$, we know that the result holds. Thus we can safely assume that both $(F_1 \cup F_2) \cup K$ and $F_2 \cup (K \otimes F_1)$ are inconsistent. Let

$P_{min}^{F_1 \cup F_2} = \min\{r(A) : F_1 \cup F_2 \vdash \neg A \text{ and } A \in K\} = \beta$

$P_{min\otimes}^{F_2} = \min\{r^1(A) : F_2 \vdash \neg A \text{ and } A \in K \otimes F_1\} = \gamma$

where r and r^1 is the rank over K and $K \otimes F_1$, respectively.

By $P_{min}^{F_1 \cup F_2} = \beta$, we know that there is $A_0 \in K$ such that $F_1 \cup F_2 \vdash \neg A_0$ and $r(A_0) = \beta$. Similarly, from $P_{min\otimes}^{F_2} = \gamma$, there is $A_1 \in K \otimes F_1$ such that $F_2 \vdash \neg A_1$ and $r^1(A_1) = \gamma$.

By $r^1(A_1) = \gamma$, we know that $P_{\leq \gamma} \cup F_1 \vdash A_1$. Then there exists a finite subset $\bar{F}_1$ of F_1 such that $\neg(\wedge \bar{F}_1) \vee A_1 \in K$(since $A_1 \in K \otimes F_1$) and $P_{\leq \gamma} \vdash \neg(\wedge \bar{F}_1) \vee A_1$. Thus $r(\neg(\wedge \bar{F}_1) \vee A_1) \leq \gamma$. On the other hand, $F_1 \cup F_2 \vdash (\wedge \bar{F}_1) \wedge \neg A_1$. By the minimality of β, $r(\neg(\wedge \bar{F}_1) \vee A_1) \geq \beta$. Therefore we have $\gamma \leq \beta$.

We are now ready to prove that $(K \otimes F_1) \otimes_{F_1} F_2 \subseteq K \otimes (F_1 \cup F_2)$. Suppose that $B \in (K \otimes F_1) \otimes_{F_1} F_2$. Then there exists $A' \in K \otimes F_1$ such that $F_2 \vdash \neg A'$ and

$$r^1(A' \vee B) < P_{min\otimes}^{F_2} \text{ or } \vdash A' \vee B \tag{3}$$

If $\vdash A' \vee B$, $F_2 \vdash B$. Thus $B \in K \otimes (F_1 \cup F_2)$. If $\not\vdash A' \vee B$, let $r^1(A' \vee B) = \delta$. Then $A' \vee B \in P_{\delta}^{F_1,0}$. It follows that $P_{\leq \delta} \cup F_1 \vdash A' \vee B$. Since $(F_1 \cup F_2) \cup K$ is inconsistent, there exists a finite subset $\bar{F}_1$ of F_1 and finite subset $\bar{F}_2$ such that $\neg(\wedge \bar{F}_1) \vee \neg(\wedge \bar{F}_2) \in K$ and $P_{\leq \delta} \vdash \neg(\wedge \bar{F}_1) \vee A' \vee B$. Hence $r(\neg(\wedge \bar{F}_1) \vee \neg(\wedge \bar{F}_2) \vee A' \vee B) \leq \delta$. Since $\delta < P_{min\otimes}^{F_2} = \gamma \leq \beta = P_{min}^{F_1 \cup F_2}$, we yield $r(\neg(\wedge \bar{F}_1) \vee \neg(\wedge \bar{F}_2) \vee A' \vee B) < P_{min}^{F_1 \cup F_2}$. By Lemma 3 we obtain that $B \in K \otimes (F_1 \cup F_2)$.

To prove that $K \otimes (F_1 \cup F_2) \subseteq (K \otimes F_1) \otimes_{F_1} F_2$, assume that $B \in K \otimes (F_1 \cup F_2)$. Then there exists $A'' \in K$ such that $F_1 \cup F_2 \vdash \neg A''$ and

$$r(A'' \vee B) < P_{min}^{F_1 \cup F_2} \text{ or } \vdash A'' \vee B \tag{4}$$

If $\vdash A'' \vee B$, then $F_1 \cup F_2 \vdash B$. It follows that there exists a finite subset $\bar{F}_2$ of F_2 such that $\neg(\wedge \bar{F}_2) \in K \otimes F_1$ and $F_1 \vdash \neg(\wedge \bar{F}_2) \vee B$. By the assumed special construction of minimal change ($\alpha = 0$), we have $r^1(\neg(\wedge \bar{F}_2) \vee B) = 0$. Since $F_1 \cup F_2$ is consistent and $P_0 \subseteq Cn(F_1)$, $P_{min\otimes}^{F_2} > 0$. Therefore $r^1(\neg(\wedge \bar{F}_2) \vee B) < P_{min\otimes}^{F_2}$. By Lemma 3 we have $B \in K \otimes F_1 \otimes_{F_1} F_2$.

In the case of $r(A'' \vee B) < P_{min}^{F_1 \cup F_2}$. Since $F_1 \cup F_2 \vdash \neg A''$, there exists a finite subset $\bar{F}_2'$ of F_2 such that $\neg(\wedge \bar{F}_2') \in K \otimes F_1$ and $F_1 \vdash \neg(\wedge \bar{F}_2') \vee A''$. Then we have $r^1(\neg(\wedge \bar{F}_2') \vee A'') = 0$. It follows that $r^1(\neg(\wedge \bar{F}_2') \vee A'' \vee B) = 0$. By the same argument above, we conclude that $B \in K \otimes F_1 \otimes_{F_1} F_2$. $\square$

Proof of Proposition 1: Assume that $A \in P_{\leq \alpha}$ and $A \notin Cn(F_1)$. Suppose that $P_{\leq \alpha} \not\subseteq Cn(F_1)$. Then there exists $B \in Cn(F_1)$ such that $B \notin P_{\leq \alpha}$. Let $F_2 = \{\neg A \vee \neg B\}$. Since $P_{\leq \alpha} \cup F_1$ is consistent, we have $P_{min}^{F_1} = \min\{r(A) : A \in K \;\&\; F_1 \vdash \neg A\} > \alpha$. It follows that $r(A) \leq \alpha < P_{min}^{F_1}$. Thus $A \in K \otimes F_1$. On the other hand, since $B \in K \otimes (F_1 \cup F_2)$, $(\otimes CN)$ implies that $B \in (K \otimes F_1) \otimes_{F_1} F_2 = (K \otimes F_1) \otimes_{F_2} \{\neg A \vee \neg B\}$. If $\vdash (A \wedge B) \vee B$, we have $\vdash B$, which contradicts $B \notin P_{\leq \alpha}$. Therefore $r^1((A \wedge B) \vee B) < r^1(A \wedge B)$, that is, $r^1(B) < r^1(A \wedge B)$, or $r^1(B) < r^1(A)$, which also contradicts $B \notin P_{\leq \alpha}$. $\square$

Proof of Theorem 3: Let $P^{F_2}_{min\otimes} = \min\{r^1(A) : F_2 \vdash \neg A \text{ and } A \in K \otimes F_1\} = \gamma$. Then there exists a sentence $A_0 \in K \otimes F_1$ such that $F_2 \vdash \neg A_0$ and $r^1(A_0) = \gamma$.

First we assume that $F_2 \cup K$ is inconsistent. Let $P^{F_2}_{min} = \min\{r(A) : F_2 \vdash \neg A \text{ and } A \in K\} = \beta$. Then there exists a sentence $A_1 \in K$ such that $F_2 \vdash \neg A_1$ and $r(A_1) = \beta$. Let $A = A_0 \vee A_1$. Then $A \in K \cap (K \otimes F_1) = K \ominus F_1$. It is not difficult to show that $\beta = r(A_1) = r(A) = r^1(A) = r^1(A_0) = \gamma$. Therefore $P^{F_2}_{min} = P^{F_2}_{min\otimes}$.

Assume that $B \in K \otimes F_2$. Then there exists $A_2 \in K$ such that $F_2 \vdash \neg A_2$ and

$$r(A_2 \vee B) < P^{F_2}_{min} \text{ or } \vdash A_2 \vee B$$

If $\vdash A \vee B$, then $B \in (K \otimes F_1) \otimes_{F_1} F_2$. Otherwise, $r(A_2 \vee B) < P^{F_2}_{min}$. Let $A' = A_0 \vee A_2$. It follows that $A' \in K \ominus F_1$ and $F_2 \vdash \neg A'$. By the construction of the minimal change, $r^1(A' \vee B) = r(A' \vee B) \leq r(A_2 \vee B) < P^{F_2}_{min} = P^{F_2}_{min\otimes}$. Thus $B \in (K \otimes F_1) \otimes_{F_1} F_2$. So $K \otimes F_1 \subseteq (K \otimes F_1) \otimes_{F_1} F_2$.

Conversely, Assume that $B \in (K \otimes F_1) \otimes_{F_1} F_2$. Then there exists $A_3 \in K \otimes F_1$ such that $F_2 \vdash \neg A_3$ and

$$r^1(A_3 \vee B) < P^{F_2}_{min\otimes} \text{ or } \vdash A_3 \vee B$$

The case of $\vdash A \vee B$ is trivial. So we only consider the other case. Let $A'' = A_1 \vee A_3$. By using the similar argument above, we can obtain $B \in K \otimes F_2$. Therefore $(K \otimes F_1) \otimes_{F_1} F_2 = K \otimes F_2$.

Now let's consider the case that $F_2 \cup K$ is consistent. Given $B \in K \otimes F_2 = K + F_2$, there exists a finite subset $\bar{F}_2$ of F_2 such that $\neg(\wedge \bar{F}_2) \in K \otimes F_2$ and $\neg(\wedge \bar{F}_2) \vee B \in K$. Thus $\neg(\wedge \bar{F}_2) \vee B \in K \ominus F_1$, which implies $r^1(\neg(\wedge \bar{F}_2) \vee B) < \eta \leq \alpha = P^{F_2}_{min\otimes}$. Thus $B \in (K \ominus F_1) \otimes_{F_1} F_2$. Conversely, if $B \in (K \ominus F_1) \otimes_{F_1} F_2$, there exists $A_4 \in K \otimes F_1$ such that $F_2 \vdash \neg A_4$ and

$$r^1(A_4 \vee B) < P^{F_2}_{min\otimes} \text{ or } \vdash A_4 \vee B$$

Then $A_4 \vee B \in K \ominus F_1 \subseteq K$. Thus $B \in K + F_2 = K \otimes F_2$. □

Proof of Proposition 2: Assume that $K \otimes F_1 \not\subseteq P^{F_1,\alpha}_{\leq\alpha}$. Then there exists $A \in K \otimes F_1$ such that $A \notin P^{F_1,\alpha}_{\leq\alpha}$. If $A \notin K$, then there exists a finite subset $\bar{F}_1$ of F_1 such that $\neg(\wedge \bar{F}_1) \vee A \in K$. If $r(\neg(\wedge \bar{F}_1) \vee A) \in P^{F_1,\alpha}_{\leq\alpha}$, then $A \in P^{F_1,\alpha}_{\leq\alpha}$, a contradiction. Therefore we can safely assume that $A \in K$. Since $F_1 \not\subseteq K$, there exists $B \in F_1$ such that $B \notin K$. Thus $A \in K + \{\neg A \vee \neg B\} = K \otimes \{\neg A \vee \neg B\}$. However, it is not hard to verify that $A \notin (K \otimes F_1) \otimes_{F_1} \{\neg A \vee \neg B\}$. This contradicts $(\otimes CB)$. □

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