Judgment Aggregation with Abstentions: a Hierarchical Approach

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Abstract. This paper presents a quasi-lexicographic judgment aggregation rule based on the hierarchy of judges. We do not assume completeness at both individual and collective levels, which means that a judge can abstain from a proposition and the collective judgment on a proposition can be undetermined. We prove that the proposed rule is (weakly) oligarchic. This is by no means a negative result. In fact, our result demonstrates that with abstentions, oligarchic aggregation is not necessarily a single level determination but can be a multiple-level democracy, which partially explains its pervasiveness in the real world.

1 Introduction

Judgment aggregation deals with the problem of how a group judgement on certain issues can be formed based on individuals' judgements on the same issues. List and Pettit presented an impossibility result which says that no aggregation rule can generate consistent collective judgments if we require the rule to satisfy a set of "plausible" conditions [1]. However, such an impossibility result does not discourage the investigation of judgement aggregation. By weakening or varying these conditions, a growing body of literature on judgement aggregation has emerged in recent years [2–6].

In our society the hierarchy is one of the most basic organization forms and a hierarchical group may give individual members or subgroups the priority to determine the collective judgments on certain propositions. However, such kind of expert rights has been rarely investigated in the current literature [7], let alone proposing a specific judgment aggregation rule to formally display how the hierarchical groups generate the collective judgments. In addition, the potential aggregation rule in the hierarchical group is likely to be oligarchic defined in [8], but the non-oligarchs still have chance to make contributions to the collective judgments on some issues. Then one of the challenges is to find the specific condition under which non-oligarchs have such power. To clarify these two questions, it is feasible to explore a plausible judgment aggregation rule for hierarchical groups.

In this paper, we focus on dealing with above questions by introducing a quasi-lexicographic approach for hierarchical groups without requiring the completeness at both the collective and individual level. This proposed aggregate procedure is proved to be oligarchic. This is by no means a negative result. In fact, our result reveals that with abstentions, oligarchic aggregation is no longer a single level determination but can also be a multiple-level democracy, which partially explains its pervasiveness in the real world.

2 The Model

The formal model of judgment aggregation with abstentions is the same as the formalism in [1, 9] except the followings: First, we restrict the agenda to a finite set of *literals* i.e., atomic propositions or their negation in the underlying logic **L**. That is $X = \{p, \neg p : p \in X^*\}$ where $X^* \subseteq \mathbf{L}$ is a set of unnegated atomic propositions. Secondly, we will not assume that each individual's judgment set $\Phi_i \subseteq X$ and the collective judgment set $\Phi \subseteq X$ must be complete. And individual i abstains from making a judgment on p, which is denoted by $p\#\Phi_i$. In other words, $p\#\Phi_i$ if and only if $p \notin \Phi_i$ and $\neg p \notin \Phi_i$. Lastly, we will assume that each individual's judgment is *individual consistence*[3]. That is, for every $p \in X$, if $p \in \Phi_i$, then $\neg p \notin \Phi_i$.

3 Conditions on Aggregation Rules

We now turn to investigating the conditions which are desirable to put on an aggregation rule in terms of abstentions. Let F be an aggregation function.

Consistency (C). *For each consistent profile* $(\Phi_i)_{i \in N} \in Dom(F)$[4], $F((\Phi_i)_{i\in N})$ *is consistent as well. That is,* $p \in F((\Phi_i)_{i\in N})$ *implies* $\neg p \notin F((\Phi_i)_{i\in N})$ *for all* $p \in X$.

Non-dictatorship (D). *There is no* $x \in N$ *such that for all* $\{\Phi_i\}_{i\in N} \in Dom(F)$, $F(\{\Phi_i\}_{i\in N}) = \Phi_x$. This is a basic democratic requirement: no single individual should always determine the collective judgment set.

Unanimity with Abstentions (U). *For every* $p \in X$ *and any* $\alpha \in \{p, \neg p\}$, *if there is some* $V \subseteq N$ *such that* $V \neq \emptyset$, $\forall i \in V.\alpha \in \Phi_i$ *and* $\forall j \in N \backslash V.\alpha\#\Phi_j$, *then* $\alpha \in F((\Phi_i)_{i\in N})$. Intuitively, if a set of individuals agree on a certain judgment on a proposition α while all the others abstain from α, then this condition requires that $F((\Phi_i)_{i\in N})$ also make the same judgment on α.

Systematicity (S). *For every* $p, q \in X$ *and every profiles* $(\Phi_i)_{i\in N}$, $(\Phi'_i)_{i\in N} \in \mathrm{Dom}(F)$, *if for every* $i \in N$, $p \in \Phi_i$ *iff* $q \in \Phi'_i$ *and* $\neg p \in \Phi_i$ *iff* $\neg q \in \Phi'_i$, *then* $p \in F((\Phi_i)_{i\in N})$ *iff* $q \in F((\Phi'_i)_{i\in N})$. This condition including independency and neutrality parts requires that propositions in the agenda should be treated in an even-handed way by the aggregation function, and the collective judgments on each proposition should depend exclusively on the pattern of individual judgment on that proposition.

[3] Given that the agenda is a set of literals, it is equivalent to logical consistence, i.e., there is a valuation v such that $v(p) = T$ for all $p \in \Phi_i$.

[4] $Dom(F)$ denotes the domain of F, i.e., the set of admissible profiles.

Note that comparing to other conditions, *Unanimity with Abstentions* is specially designed for judgments with abstentions, and *non-dictatorship* can be derived from it.

Proposition 1. *Every judgment aggregation rule satisfying unanimity with abstentions is non-dictatorial.*

In the following, we denote *Consistency*, *Unanimity with Abstentions* and *Systematicity* as **CUS** for short.

4 The Quasi-Lexicographic Aggregation Rule

In this section, we define an aggregation rule that satisfies the above-mentioned three fundamental conditions. Firstly we need two auxiliary concepts, which are precisely defined as follows:

Definition 1. *Let N be a set of individuals, $(N, <)$ is called a* hierarchy *over N if $<$ satisfies transitivity and asymmetry.*

Definition 2. *Let $<$ be a hierarchy over N. For every $p \in X$, p is* not collectively rejected by aggregation rule F, *denoted by $p \rhd F((\Phi_i)_{i\in N})$, if there is an individual with greater priority accepting it once it is rejected by some individual. That is,*

$$p \rhd F((\Phi_i)_{i\in N}) \text{ iff } \forall i \in N(\neg p \notin \Phi_i \vee \exists j \in N(i < j \wedge p \in \Phi_j)) \quad (1)$$

We denote the negation of $p \rhd F((\Phi_i)_{i\in N})$ as $\overline{p \rhd F((\Phi_i)_{i\in N})}$, meaning that p is neither collectively accepted nor collectively undetermined. Based on the concept of "non-rejection" $\rhd$, we can define an aggregate procedure F for that *p is collectively accepted*, denoted by $p \in F((\Phi_i)_{i\in N})$, as follows.

Definition 3. *For all $p \in X$,*

$$p \in F((\Phi_i)_{i\in N}) \text{ iff } p \rhd F((\Phi_i)_{i\in N}) \text{ and } \exists j \in N.p \in \Phi_j \quad (2)$$

Intuitively, this aggregate procedure says that p is accepted by a group if p is not collectively rejected and there is at least one individual who accepts it.

Obviously, a proposition p is *collectively undetermined* is decided by the following condition:

$$p \# F((\Phi_i)_{i\in N}) \text{ iff } p \notin F((\Phi_i)_{i\in N}) \text{ and } \neg p \notin F((\Phi_i)_{i\in N}). \quad (3)$$

We will call the above defined rule F *the quasi-lexicographic rule*. And the next proposition shows that the quasi-lexicographic rule F satisfies the desirable conditions **CUS**.

Proposition 2. *The quasi-lexicographic rule F satisfies conditions* **CUS**.

The following theorem has double value: on one hand, it is helpful to understand the quasi-lexicographic rule more intuitively; on the other hand, it is useful to prove the main result.

Theorem 1. *Given that* $(N,<)$ *is well-prioritized*[5], *and let* $\Phi = F((\Phi_i)_{i\in N})$. *Then*

1. $p \rhd \Phi$ *iff* $\forall i \in N((\forall j > i.p\#\Phi_j) \to \neg p \notin \Phi_i)$.
2. $p \rhd \Phi$ *iff* $\forall i \in N(\neg p \notin \Phi_i \vee (\exists j > i.p \in \Phi_j \wedge \forall j' > j.p\#\Phi_{j'}))$.

In the last part of this section, we will show that the quasi-lexicographic rule F is weakly oligarchic [6] and displays nicely under what conditions the non-oligarchs can have the power to affect the collective decision-making.

Proposition 3. *The quasi-lexicographic rule* F *is weakly oligarchic, but not strictly oligarchic.*

In order to reveal how the non-oligarchs can have the power to make collective decisions through F, we need two further definitions.

Definition 4. *A set of judges* D *is decisive on* $p \in X$ *for a judgment aggregation function* G *iff for every profile* $(\Phi_i)_{i\in N} \in Dom(G)$, *if* $p \in \Phi_j$ *for every* $j \in D$, *then* $p \in G((\Phi_i)_{i\in N})$.

Definition 5. *Given a hierarchy on* N *and induced by* $<$, N *can be divided into subgroups* $M_1, \cdots, M_n$, *where* $\emptyset \neq M_i \subseteq N$ *for every* $i \in N$, $\bigcup_{i=1}^{n} M_i = N$ *and* M_i *is inductively defined as follows:*

- $M_1 = \{i \in N : \nexists j \in N.i < j\}$
- $M_{k+1} = \{i \in N\backslash(\bigcup_{i=1}^{k} M_i) : \nexists j \in N\backslash(\bigcup_{i=1}^{k} M_i).i < j\}$

We finally come to the following result, which displays that a proposition is not rejected by every of the superiors is sufficient and necessary to make the subgroup composed of the immediate inferiors a decisive set on this proposition.

Theorem 2. *Given a hierarchy on* N, *and let* $M_1, \cdots, M_n$ *be the subgroups of each level, for every* $k \in \{1, \cdots, n\}$ *and* $p \in X$, M_k *is decisive on* p *for the quasi-lexicographic rule* F *if and only if* $\neg p \notin \Phi_i$ *for every* $i \in \bigcup_{h=0}^{k-1} M_h$.

This follows that non-oligarchs can have the power to make collective decisions on some proposition if and only if the proposition is not rejected by the oligarchs, which displays that with abstentions, oligarchic aggregation is no longer a single level determination but can also be a multiple-level democracy.

[5] If there is no infinite ascending sequence $i_1 < i_2 < i_3 < \cdots$, where $i_n \in N$, which is automatically satisfied by every hierarchy over N

[6] Refer to [10] for its precise definition.

5 Conclusion

In this paper, we have investigated the aggregation rules for judgment aggregation without requiring the completeness at both individual and collective levels. Different from the perspective in [7] which presents the first extension of Sen's liberal paradox, we focus on dealing with two questions: How does the hierarchical group generate the collective judgments? How can the non-oligarchs have the power to make the collective decision in an oligarchic environment? We have replied them by proposing a quasi-lexicographic rule for hierarchical groups, which is inspired by [11]. This judgment aggregation rule turns out to satisfy the desirable conditions and reveal that if certain issues are not rejected by the oligarchs, then non-oligarchs have the decisive power in making group decisions on these issues. This seems positive news to the result in [10], since this rule demonstrates that with abstentions, even an oligarchic aggregation procedure can also realize a multiple-level democracy. To some extent, this also explains why hierarchical (oligarchic) systems exist widely in the real world.

Acknowledgments. We are grateful to Laurent Perrussel and Meiyun Guo for valuable suggestions, and special thanks are due to three anonymous referees for their insightful comments. This research was partially supported by the Australian Research Council through Discovery Project DP0988750.

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