Some Contributions to Nonmonotonic Consequence

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Abstract This paper introduces a non-Horn rule **WRM** which is a weak form of *rational monotony*. We explore the effects of adding this non-Horn rule to the rules for the preferential inference. In this paper, a relation $|\!\sim$ is said to be $\mathbf{P+WRM}$ iff it is a *preferential* inference and satisfies the rule **WRM**. We establish the representation theorem for $\mathbf{P+WRM}$, and compare the strength of **WRM** with some non-Horn rules appearing in literatures. Moreover, we explore the relation between $\mathbf{P+WRM}$ and conditional logic, and demonstrate that $\mathbf{P+WRM}$ is equivalent to 'flat' fragment of conditional logic **CS4.2**. Another contribution of this paper is to explore the relation between two special kinds of preferential models, i.e., PRC model and quasi-linear model. Main result reveals that the latter is a special form of the former.

Keywords nonmonotonic consequence relation, conditional logic, rational monotony, injective PRC model, quasi-linear model

1 Introduction

It is widely acknowledged that commonsense reasoning is nonmonotonic, or defeasible. Many researchers have proposed systems that perform such nonmonotonic inferences. The best known are probably: negation as failure, circumscription, the modal system of McDermott and Doyle, default logic and autoepistemic logic. Each of those systems is worth studying by itself, but a general framework in which those examples could be compared and classified is missing. Gabbay was probably the first to suggest focusing the study of nonmonotonic logic on their inference relations[1]. He proposed a generalization of logical inference, motivated by default reasoning, in which monotony is violated for the purposes of representing the behavior of nonmonotonic inference systems as inference relations. Gabbay suggested three basic conditions that any such relation should meet *reflexivity, cut* and *weak* monotony (i.e., *cautious* monotony in [2]). Gabbay argued for his three conditions on proof-theoretic grounds but provided no semantics against which to check them. Recently, inspired by Gabbay's work, many people have researched into abstract nonmonotonic inference relation from various angles[2−12].

Among them, Lehman, Magidor and others have investigated the effects of adding the non-Horn rule called *rational monotony* to the rules for preferential inference presented in [4]. They think that *rational monotony* is desirable. However, David Makinson points out that[2] *rational monotony* is too strong to insist upon and we should explore other more appealing way of constructing nonmonotonic inference relations with the desired behavior:

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validation of *cumulativity, supraclassicality, distribution* and *disjunctive rationality* but not in general of *rational monotony*.

Bezzazi and Pino Pérez begin a semantic investigation on two other non-Horn rules, called *rational transitivity* and *rational contraposition*[10]. Bezzazi, Makinson and Pino Pérez study these and their related conditions more systematically, and establish interrelations and provide semantic characterizations in [8]. In particular, they establish the representation theorem for $\mathbf{P}+\mathbf{RM}+\mathbf{RC}$ and $\mathbf{P}+\mathbf{RM}+\mathbf{WD}$ (see Section 2 for the definition of these systems), however, their result leave open the question of representation theorems for the weaker postulate sets $\mathbf{P}+\mathbf{WD}$ and $\mathbf{P}+\mathbf{RC}$. In [9], we introduce valuation-ranked model and PRC model, and solve these questions in the framework of finite propositional logic.

This paper introduces a non-Horn rules **WRM** (Weak Rational Monotony) that is a weak form of *rational monotony*, and explore the effects of adding this non-Horn rule to the rules for preferential inference. A relation $|\!\sim$ is said to be $\mathbf{P}+\mathbf{WRM}$ iff it is *preferential* inference and satisfies the rule **WRM**. We establish the representation theorem for $\mathbf{P}+\mathbf{WRM}$, and compare the strength of **WRM** with some non-Horn rules appearing in literatures. We also explore the relation between $\mathbf{P}+\mathbf{WRM}$ and conditional logic, and demonstrate that $\mathbf{P}+\mathbf{WRM}$ is equivalent to 'flat' fragment of conditional logic **CS4.2**.

Moreover, this paper explores the relation between PRC model[9] and quasi-linear model[8], and establish another representation theorem for the nonmonotonic inference relation $\mathbf{P}+\mathbf{RT}$.

2 Preliminaries

In this section, we will recall some basic definitions and results from [4, 5, 8], which will be used in this paper.

2.1 Some Horn and Non-Horn Rules

We consider formulae of classical propositional calculus built over a set of atomic formulae denoted by L plus two constants $\top$ and $\bot$ (the formulae *true* and *false* respectively). Let $Form(L)$ be the set of all well-formed formulae. If L is finite, we will say that the propositional language is finite. Let U be the set of valuations, i.e., functions v: $L\cup\{\top,\bot\}\to\{0,1\}$ such that $v(\top)=1$ and $v(\bot)=0$. We use lower case letters of the Greek alphabet to denote formulae, and the letters $v, v_1, v_2\ldots$, to denote valuations. As usual, $\vdash\alpha$ means that α is a tautology and $v\models\alpha$ means that v satisfies α where compound formulae are evaluated as usual.

We consider certain binary relations between formulae. These relations will be called inference relations or consequence relations. Gabbay uses the relation symbol $|\!\sim$ to denote nonmonotonic consequence to distinguish it from monotonic logical consequence. If α,β are formulae, then the sequence $\alpha\,|\!\sim\beta$ is called a conditional assertion. In [1], a consequence relation is defined as any binary relation R between propositional formulae for which certain properties hold. If a pair $(\alpha,\beta)\in R$, then using this notion of consequence, one may sensibly conclude β given α, and write $\alpha\,|\!\sim\beta$. $\alpha\,|\!\not\sim\beta$ means $(\alpha,\beta)\notin R$. Certain especially interesting properties of sets of conditional assertions are described as follows, the intuition behind those rules may be found in [2, 4, 5, 6, 8].

LLE (Left Logical Equivalence)

$$\frac{\models\alpha\leftrightarrow\beta,\alpha\,|\!\sim\gamma}{\beta|\sim\gamma}$$

RW (Right Weakening)

$$\frac{\models\alpha\to\beta,\gamma\,|\!\sim\alpha}{\gamma\,|\!\sim\beta}$$

Or

$$\frac{\alpha\,|\!\sim\gamma,\beta\,|\!\sim\gamma}{\alpha\vee\beta\,|\!\sim\gamma}$$

And

$$\frac{\alpha\,|\!\sim\beta,\alpha\,|\!\sim\gamma}{\alpha\,|\!\sim\beta\wedge\gamma}$$

CM (Cautious Monotony)

$$\frac{\alpha \mid\sim \beta, \alpha \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma}$$

Cut

$$\frac{\alpha \wedge \beta \mid\sim \gamma, \alpha \mid\sim \beta}{\alpha \mid\sim \gamma}$$

NR (Negation Rationality)

$$\frac{\alpha \wedge \gamma \mid\not\sim \beta, \alpha \wedge \neg\gamma \mid\not\sim \beta}{\alpha \mid\not\sim \beta}$$

DR (Disjunctive Rationality)

$$\frac{\alpha \mid\not\sim \beta, \gamma \mid\not\sim \beta}{\alpha \vee \gamma \mid\not\sim \beta}$$

RM (Rational Monotony)

$$\frac{\alpha \wedge \gamma \mid\not\sim \beta, \alpha \mid\not\sim \neg\gamma}{\alpha \mid\not\sim \beta}$$

DP (Determinacy Preservation)

$$\frac{\alpha \wedge \gamma \mid\not\sim \neg\beta, \alpha \mid\sim \beta}{\alpha \wedge \gamma \mid\sim \beta}$$

RT (Rational Transitivity)

$$\frac{\alpha \mid\sim \beta, \beta \mid\sim \gamma, \alpha \mid\not\sim \neg\gamma}{\alpha \mid\sim \gamma}$$

RC (Rational Contraposition)

$$\frac{\alpha \mid\sim \beta, \neg\beta \mid\not\sim \alpha}{\neg\beta \mid\sim \neg\alpha}$$

WD (Weak Determinacy)

$$\frac{\alpha \mid\not\sim \beta, \top \mid\sim \neg\alpha}{\alpha \mid\sim \neg\beta}$$

Reflexivity

$$\alpha \mid\sim \alpha$$

M (Monotony)

$$\frac{\alpha \mid\sim \beta}{\alpha \wedge \gamma \mid\sim \beta}$$

An inference relation $\mid\sim$ is said to be *cumulative*[4] iff it contains all instances of **Reflexivity** axiom and is closed under the inference rules of **LLE, RW, Cut** and **CM**. We shall name this system **C** for *cumulative*①. The system **P**[4] consists of all the rules of **C** and the rule **Or**. An inference relation that satisfies all the rules of **P** is said to be *preferential*.

Lehmann and Magidor introduce *rational* inference relation in [5]. The system **R** consists of all the rules of **P** and the rule **RM**. An inference relation that satisfies all the rules of **R** is said to be a *rational* inference *relation*. In this paper, we introduce a new non-Horn rule named **WRM** (Weak Rational Monotony) as follows:

$$\frac{\alpha \mid\sim \gamma, \alpha \wedge \beta \mid\not\sim \gamma, \alpha \mid\not\sim \neg\beta}{\top \mid\sim \neg\alpha}$$

It is easy to see that the rule **WRM** restricts the employment of the rule **RM** through strengthening the upper sequents of **RM**. The system **P + WRM** consists of all the rules of **P** and **WRM**. An inference relation that satisfies all the rules of **P** and the rule **WRM** is said to be a **P + WRM** inference relation.

2.2 Preferential Model

Following the definition in [4], a *preferential* model W is a triple $\langle S, l, \prec\rangle$, where S is a set, the elements of which will be called states, the interpretation function $l : S \to U$ assigns a valuation to each state, where U is the set of all valuations, and $\prec$ is a strict partial order on S (i.e., $\prec$ is transitive and irreflexive) satisfying the following smoothness condition: for any $\alpha \in Form(L)$, the set $\parallel \alpha \parallel_w = \{s : s \in W \text{ and } l(s) \models \alpha\}$ is smooth②. If there is no ambiguity, we shall write $\parallel \alpha \parallel$ instead of $\parallel \alpha \parallel_w$. A preferential model $W = \langle S, l, \prec\rangle$ is said to be injective model iff l is injective.

①Notice that **AND** is a drived rule of system **C** (see [4]). In the following, we will use it without proof.

②Let W be a set, $\prec$ be a strict partial order over W and $V \subseteq W$, we shall say that V is smooth iff for any $t \in V$, either t is itself minimal in V (i.e., there is no $w \in V$ such that $w \prec t$), or there exists $s \in V$ such that $s \prec t$ and s is minimal in V.

Let $W = \langle S, l, \prec \rangle$ be a *preferential model*. We adopt the following notations: the range of l will be denoted by $rang(l)$ (i.e., $rang(l) =_{\text{def}} \{v : \exists s\ (s \in S \text{ and } l(s) = v)\}$). If $X \subseteq S$, then $\min(X)$ is the set of all minimal element of X with respect to $\prec$ (i.e., $\min(X) =_{\text{def}} \{t \in X : \neg\exists s\ (s \in X \text{ and } s \prec t)\}$), $l(X) =_{\text{def}} \{v : \exists s\ (s \in X \text{ and } l(s) = v)\}$. If $\Sigma \subseteq rang(l)$, then $l^{-1}(\Sigma) =_{\text{def}} \{s \in S : \exists v\ (v \in \Sigma \text{ and } l(s) = v)\}$, we shall write $l^{-1}(v)$ instead of $l^{-1}(\{v\})$.

A *ranked* model $W = \langle S, l, \prec \rangle$ is a preferential model for which the strict partial order $\prec$ is modular, i.e., for any $x, y, z \in S$, if $x \not\prec y$, $y \not\prec x$ and $z \prec x$, then $z \prec y$.

Let $W = \langle S, l, \prec \rangle$ be a *preferential* model, the inference relation generated by W will be denoted by $|\!\sim_w$ and is defined as follows: for any formulae α and β, $\alpha \mid\!\sim_w \beta$ iff for any s minimal in $\parallel \alpha \parallel$, $l(s) \models \beta$. We denote the set $\{\beta : \alpha \mid\!\sim_w \beta\}$ by $C_w(\alpha)$. An inference relation $|\!\sim$ is said to be injective preferential relation iff there exists an injective preferential model W such that $|\!\sim = |\!\sim_w$.

One of the main tools in the study of nonmonotonic inference relations is the representation of such relations in terms of preferential models. Lehmann, Magidor and others have investigated the semantic characterization of preferential relation and rational relation in [4, 5]. In particular, they have established the representation theorems for them respectively.

Theorem 2.1[4,5]**.** *$|\!\sim$ is a preferential inference relation iff there is a preferential model $W = \langle S, l, \prec \rangle$ such that $|\!\sim = |\!\sim_w$.*

$|\!\sim$ is a rational inference relation iff there is a ranked model $W = \langle S, l, \prec \rangle$ such that $|\!\sim = |\!\sim_w$.

In order to establish a representation theorem for **P + RT** (i.e., **P + RM + RC**, **P + RM + WD**), Bezzazi, Makinson and Pino Pérez introduce quasi-linear model as follows.

Definition 2.1[8]**.** *A preferential model $W = \langle S, l, \prec \rangle$ is said to be quasi-linear iff it is ranked and it has at most one state at any level above the lowest. In other words quasi-linear means ranked and whenever $r \prec s, r \prec t$ then $s = t$ or $s \prec t$ or $t \prec s$.*

The following (representation) theorem is due to Bezzazi, Makinson and Pino Pérez.

Theorem 2.2[8]**.** *The following conditions are equivalent for any preferential inference relation $|\!\sim$:*

(1) $|\!\sim$ *is generated by some quasi-linear model.*

(2) $|\!\sim$ *is* **determinacy preserving.**

(3) $|\!\sim$ *is* **rational transitive**.

(4) $|\!\sim$ *satisfies both* **RM** *and* **RC.**

(5) $|\!\sim$ *satisfies both* **RM** *and* **WD**.

3 WRM Model

In order to establish a representation theorem for **P + WRM***, we introduce* **WRM** *model as follows.*

Definition 3.1. *A preferential model $\langle S, \ell, \prec \rangle$ is said to be a* **WRM** *model iff the relation $\prec$ satisfies the following condition: there exist a strict partial order set $\langle \Omega, < \rangle$ with minimum*③ *and a surjection function $\gamma : S \to \Omega$ such that for any $s, t \in S$, $\gamma(s) < \gamma(t)$ iff $s \prec t$.*

Lemma 3.1. *A preferential model $\langle S, \ell, \prec \rangle$ is a* **WRM** *model iff for any $s \in S$ and $w \in \min(S)$, if $s \notin \min(S)$ then $w \prec s$.*

Proof. ($\Rightarrow$) Since $\langle S, \ell, \prec \rangle$ is a **WRM** model, there exist a strict partial order set $\langle \Omega, < \rangle$ with minimum and a surjection function $\gamma : S \to \Omega$ such that for any $s, t \in S$, $\gamma(s) < \gamma(t)$ iff $s \prec t$. We denote the minimum by w_0. It is easy to show that, for any $w \in S$, $w \in \min(S)$

③Let W be a set, $\prec$ be a partial order on W and $V \subseteq W$, we shall say that $t \in V$ is a minimum of V iff for every $s \in V$, $s \neq t$, we have $t \prec s$. If t is a minimum of W, we also say that t is the minimum of $\prec$.

iff $\gamma(w) = w_0$. Suppose that w, $s \in S$, $s \notin \min(S)$ and $w \in \min(S)$. Thus, we get $w_0 = \gamma(w) < \gamma(s)$. Since $\gamma(w) < \gamma(s)$ iff $w \prec s$, we obtain $w \prec s$.

($\Leftarrow$) Constructing a partial order set $\langle \Omega, < \rangle$ and a surjection function $\gamma : S \to \Omega$ as follows:

(1) $\Omega = (S - \{w : w \in \min(S)\}) \cup \{s_0\}$, where $s_0 \notin S$;

(2) $<=\prec\Downarrow (S - \{w : w \in \min(S)\}) \cup \{\langle s_0, t\rangle : t \notin \min(S)\}$; ④

(3) For any $s \in S$, $\gamma(s) = \begin{cases} s, & \text{if } s \notin \min(S); \\ s_0, & \text{if } s \in \min(S). \end{cases}$

From the above construction, it is easy to show that $\langle \Omega, < \rangle$ is a strict partial order set with the minimum s_0 and the surjection function $\gamma : S \to \Omega$ satisfy that, for any $s, t \in S$, $\gamma(s) < \gamma(t)$ iff $s \prec t$. Hence, $\langle S, \ell, \prec \rangle$ is a **WRM** model, as desired. □

Theorem 3.1. *If $W = \langle S, \ell, \prec \rangle$ is a* **WRM** *model, then $|\!\sim_w$ is a* **P + WRM** *inference relation.*

Proof. It is enough to show that $|\!\sim_w$ satisfies **WRM**. We proceed by reduction to absurdity. Suppose that the relation $|\!\sim_w$ does not satisfy **WRM**. Thus, there exist A, B and C such that $A |\!\sim_w C$, $A \wedge B \not|\!\sim_w C$, $A \not|\!\sim_w \neg B$ and $\top \not|\!\sim_w \neg A$. From $\top \not|\!\sim_w \neg A$, we conclude that there is a minimal element in S that satisfies A. Let $t_1 \in S$ be such a state. The assumption $A |\!\sim_w C$ enables us to conclude that $\ell(t_1) \models C$. From $A \not|\!\sim_w \neg B$, we know that there exists a minimal element in $\| A \|$ that satisfies B. Let $t_2 \in S$ be such a state. Thus, $\ell(t_2) \models C$ holds by the assumption $A |\!\sim_w C$. Similarly, by the assumption $A \wedge B \not|\!\sim_w C$, there exists a minimal element t_3 in $\| A \wedge B \|$ such that $\ell(t_3) \models \neg C$. Since $A |\!\sim_w C$, t_3 is not minimal in $\| A \|$. We consider two cases.

Case 1. Suppose that t_2 is minimal in S. By Lemma 3.1, we obtain $t_2 \prec t_3$. It contradicts that t_3 is minimal in $\| A \wedge B \|$.

Case 2. Suppose that t_2 is not minimal in S. Since t_1 is minimal in S, by Lemma 3.1, we get $t_1 \prec t_2$. It contradicts that t_2 is minimal in $\| A \|$. □

4 Interrelations Between WRM and Other Non-Horn Rules

In this section, we compare the strength of the rule **WRM** with some non-Horn rules appearing in literatures.

Observation 4.1. P$\not\Rightarrow$ WRM.

Proof. It is enough to construct a preferential model in which **WRM** does not hold. Let L be the propositional calculus on the three variables p_1, p_2 and p_3. Let $S = \{t_1, t_2, t_3, t_4\}$, $\prec = \{\langle t_3, t_4 \rangle\}$, $\ell(t_1) = \{p_1, p_3\}$ ⑤, $\ell(t_2) = \{p_1, p_2, p_3\}$, $\ell(t_3) = \{p_1, p_3\}$ and $\ell\ (t_4) = \{p_1, p_2\}$. This defines a preferential model W. It is easy to verify that $p_1 |\!\sim_w p_3$, $p_1 \wedge p_2 \not|\!\sim_w p_3$, $p_1 \not|\!\sim_w \neg p_2$ and $\top \not|\!\sim_w \neg p_1$. Hence, $|\!\sim_w$ does not satisfy **WRM**. □

Observation 4.2. P + WRM $\not\Rightarrow$ RM.

Proof. We shall build a **WRM** model that generates an inference relation satisfying **WRM** but not **RM**. Let L be the propositional calculus on the three variables: p_1, p_2 and p_3. $S = \{t_1, t_2, t_3, t_4\}$, $\prec = \{\langle t_1, t_2\rangle, \langle t_1, t_3\rangle, \langle t_1, t_4\rangle, \langle t_3, t_4\rangle\}$, $\ell(t_1) = \{p_2, p_3\}$, $\ell(t_2) = \{p_1, p_2, p_3\}$, $\ell(t_3) = \{p_1, p_3\}$, $\ell(t_4) = \{p_1, p_2\}$. This defines a **WRM** model W. By Theorem 3.1, $|\!\sim_w$ satisfies **WRM**. It is easy to verify that $p_1 |\!\sim_w p_3$, $p_1 \not|\!\sim_w \neg p_2$ and $p_1 \wedge p_2 \not|\!\sim_w p_3$. So, $|\!\sim_w$ does not satisfy **RM**. □

Observation 4.3. P + WRM $\not\Rightarrow$ DR.

Proof. Let us consider the following **WRM** model W. The model W has five states s_i $(0 \le i \le 4)$, the ordering is: $s_0 \prec s_1$, $s_1 \prec s_2$, $s_0 \prec s_2$, $s_0 \prec s_3$, $s_0 \prec s_4$ and $s_4 \prec s_3$. The

④ $R \Downarrow A$ is the restriction relation of R with respect to A, i.e., $R \Downarrow A = R \cap A^2$.

⑤ We give the valuations as for a Herbrand model, that is identifying the subset of variables with its characteristic function.

language has three propositional variables: p, q and r. State s_0 is labeled with the valuation that does not satisfy any variables. State s_1 is labeled with the valuation that satisfies only p and r. The two states s_2 and s_3 are labeled with the same valuation that satisfies only p and q. State s_4 is labeled with the valuation that satisfies only q and r. Since W is a **WRM** model, $|\sim_w$ satisfies **WRM**. On the other hand, it is easy to verify that $p \vee q \mid\sim_w r$, $p \mid\not\sim_w r$ and $q \mid\not\sim_w r$. So, $|\sim_w$ does not satisfy **DR**. □

Observation 4.4[5]**.** $\mathbf{P} + \mathbf{DR} \Rightarrow \mathbf{NR}$.

Observation 4.5. $\mathbf{P} + \mathbf{DR} \not\Rightarrow \mathbf{WRM}$.

Proof. Let L be the propositional calculus on the three variables: p_0, p_1 and p_2. Let S contain three elements: s_i for $i = 0, 1, 2$ and $\ell(s_i)$ satisfies only p_i. The partial order $\prec = |\langle s_1, s_2\rangle|$. This defines a preferential model W. First, we show that $|\sim_w$ does not satisfy **WRM**. Indeed, we have $p_0 \vee p_1 \vee p_2 \mid\sim_w \neg p_2$, $p_0 \vee p_1 \vee p_2 \mid\not\sim_w \neg\neg p_1$ and $\top \mid\not\sim_w \neg(p_0 \vee p_1 \vee p_2)$, nevertheless, we also have $\neg p_1 \wedge (p_0 \vee p_1 \vee p_2) \mid\not\sim_w \neg p_2$. Hence, $|\sim_w$ does not satisfy **WRM**. On the other hand, by the proof of Lemma 3.6 in [5], we know that any preferential model that does not satisfy **DR** must have at least four states. Thus, $|\sim_w$ satisfies **DR**. □

Observation 4.6. $\mathbf{P} + \mathbf{NR} \not\Rightarrow \mathbf{WRM}$.

Proof. Immediately from Observations 4.4 and 4.5. □

Observation 4.7. $\mathbf{P} + \mathbf{WRM} \not\Rightarrow \mathbf{NR}$.

Proof. Let L be the propositional calculus on the three variables: p_0, p_1 and p_2. Let S contain six elements: s_i for $i = 0..5$, and ℓ such that $\ell(s_0) \models \neg(p_0 \vee p_1 \vee p_2)$, $\ell(s_1) \models p_0 \wedge p_1 \wedge \neg p_2$, $\ell(s_2) \models p_0 \wedge \neg p_1 \wedge p_2$, $\ell(s_3) \models p_0 \wedge p_1 \wedge p_2$, $\ell(s_4) \models p_0 \wedge \neg p_1 \wedge \neg p_2$ and $\ell(s_5) \models \neg p_0 \wedge \neg p_1 \wedge \neg p_2$. $\prec = \{\langle s_0, s_i\rangle\colon 1 \le i \le 4\} \cup \{\langle s_5, s_i\rangle\colon 1 \le i \le 4\} \cup \{\langle s_2, s_1\rangle, \langle s_3, s_4\rangle\}$. This defines a **WRM** model W. It is easy to verify that $p_0 \wedge p_1 \mid\not\sim_w p_2$, $p_0 \wedge \neg p_1 \mid\not\sim_w p_2$ and $p_0 \mid\sim_w p_2$. Hence, $|\sim_w$ does not satisfy **NR**. □

Bezzazi, Makinson and Pino Pérez study some non-Horn rules systematically, and establish interrelations and provide semantic characterizations in [8]. They compare the strength of the rules **DP, RT, RC, WD** and **RM**, and prove the following proposition:

Proposition 4.1[8]**.** *Given the preferential rules* **P***, the rules* **DP** *and* **RT** *are equivalent, and are implied by* **M***. They are also equivalent to the pair* {**RM***,* **RC**} *and also to the pair* {**RM***,* **WD**}*. Moreover, given* **P***,* **RC** *implies both* **WD** *and* **NR***. However, given* **P***, none of the following implications hold:* **RM** *to* **WD***,* **RC** *to* **DR***,* **WD** *to* **NR***.*

They also establish the representation theorem for $\mathbf{P} + \mathbf{RT}$. However, their result leave open the question of representation theorems for the weaker postulate sets $\mathbf{P} + \mathbf{WD}$ and $\mathbf{P} + \mathbf{RC}$. In [9], we solve these questions in the framework of finite propositional logic. In order to compare the strength of **WRM** with **WD** and **RC**, we recall some concepts and result in [9].

Definition 4.1[9]**.** *Let $W = \langle S, l, \prec\rangle$ be a preferential model, the binary relation $\sqsubset$ is defined as follows*:

For any $X_1, X_2 \subseteq S, X_1 \sqsubset X_2$ iff $\forall s\ (s \in X_2 \Rightarrow \exists t\ (t \in X_1$ and $t \prec s))$.

Definition 4.2[9]**.** *A preferential model $W = \langle S, l, \prec\rangle$ is said to be valuation-ranked iff $\sqsubset\Downarrow \{l^{-1}(v)\colon v \in rang(l) - l(\min(S))\}$ is a linear order. In other words, for any $v_1, v_2 \in rang(l) - l(\min(S))$, if $v_1 \ne v_2$ then $l^{-1}(v_1) \sqsubset l^{-1}(v_2)$ or $l^{-1}(v_2) \sqsubset l^{-1}(v_1)$.*

Theorem 4.1[9]**.** *In finite framework, $|\sim$ is a* $\mathbf{P} + \mathbf{WD}$ *inference relation if and only if there is a valuation-ranked preferential model $W = \langle S, l, \prec\rangle$ such that $|\sim = |\sim_w$.*

Observation 4.8. (1) $\mathbf{P} + \mathbf{WRM} \not\Rightarrow \mathbf{RC}$. (2) $\mathbf{P} + \mathbf{RC} \not\Rightarrow \mathbf{WRM}$. (3) $\mathbf{P} + \mathbf{WRM} \not\Rightarrow \mathbf{WD}$. (4) $\mathbf{P} + \mathbf{WD} \not\Rightarrow \mathbf{WRM}$.

Proof. By Proposition 4.1, we have $\mathbf{P} + \mathbf{RM} \not\Rightarrow \mathbf{RC}$ and $\mathbf{P} + \mathbf{RM} \not\Rightarrow \mathbf{WD}$. Thus, it is easy to know that $\mathbf{P} + \mathbf{WRM} \not\Rightarrow \mathbf{RC}$ and $\mathbf{P} + \mathbf{WRM} \not\Rightarrow \mathbf{WD}$. In the following, we will show $\mathbf{P} + \mathbf{WD} \not\Rightarrow \mathbf{WRM}$.

Let L be the propositional calculus on the three variables: p_0, p_1 and p_2. Let S contain six elements: s_i for $i = 0, 1, 2, \ldots, 5$, and ℓ such that $\ell(s_0) \models \neg p_0 \wedge p_1 \wedge \neg p_2$, $\ell(s_1) \models p_0 \wedge p_1 \wedge p_2$,

$\ell(s_2) \models p_0 \wedge \neg p_1 \wedge p_2$, $\ell(s_3) \models p_0 \wedge p_1 \wedge \neg p_2$, $\ell(s_4) \models p_0 \wedge p_1 \wedge \neg p_2$ and $\ell(s_5) \models p_0 \wedge p_1 \wedge p_2$. $\prec = \{\langle s_0, s_1\rangle, \langle s_2, s_3\rangle, \langle s_2, s_4\rangle\}$. This defines a *valuation-ranked* preferential model W. Hence, by Theorem 4.1, $|\!\sim_w$ is a $\mathbf{P}+\mathbf{WD}$ inference relation. On the other hand, one easily verifies that $p_0 \wedge p_1 \not|\!\sim_w p_2$, $p_0 \not|\!\sim_w \neg p_1$, $p_0 \;|\!\sim_w p_2$ and $\top \not|\!\sim \neg p_0$. Thus, $|\!\sim_w$ does not satisfy **WRM**. Similarly, according to the representation theorem for $\mathbf{P}+\mathbf{RC}$ in [9], we may show $\mathbf{P}+\mathbf{RC} \not\Rightarrow$ **WRM**. □

Together with the results in [2] that **M** implies **DP** but not conversely, and that **RM** implies **DR** which implies **NR** but neither conversely. The above results are shown in Fig.1, where one condition implies another, given a preferential inference relation, iff one can follow arrows from the former to the latter.

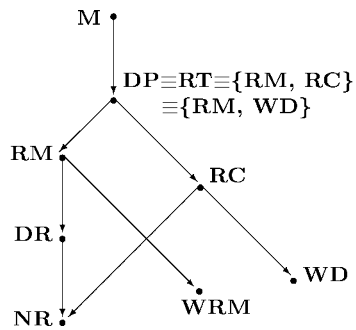

Fig.1. The strength of some rules.

5 WRM-Transform and Representation Theorem for P + WRM

In this section, we establish the representation theorem for $\mathbf{P}+\mathbf{WRM}$. In the proofs for representation theorem, it is usually necessary to construct a preferential model for a given consequence relation. In various proofs for representation theorems, the authors usually construct desired model directly based on given inference relation[4−6,8,11]. In this paper, we do not construct desired model directly, but transform a KLM model to a **WRM** model, where KLM model is introduced by Kraus *et al.* in [4]. We first recall some concepts and results in [4].

Let $|\!\sim$ be a preferential inference relation, following the technique in [4], we say that α is not less ordinary than β and write $\alpha \leq \beta$ iff $\alpha \vee \beta \;|\!\sim \alpha$. The valuation $m \in U$ is said to be a normal valuation for α (i.e., *α-normal*) iff $\forall \beta \in L$ such that $\alpha \;|\!\sim \beta$, $m \models \beta$. Given a preferential relation $|\!\sim$, Kraus *et al.* construct the preferential model $W_c = \langle S_c, \ell_c, \prec_c\rangle$ as follows:

(1) $S_c = \{\langle m, \alpha\rangle : m$ is a normal valuation for $\alpha\}$,

(2) $\ell_c\ (\langle m, \alpha\rangle) = m$, and

(3) $\langle m, \alpha\rangle \prec_c \langle n, \beta\rangle$ iff $\alpha \leq \beta$ and $m \models \neg\beta$.

For convenience, in the following, we call the above model $W_c = \langle S_c, \ell_c, \prec_c\rangle$ KLM model associated with the relation $|\!\sim$.

Lemma 5.1[4]**.** *If $\alpha \leq \beta$ and m is a α-normal valuation that satisfies β, then m is β-normal.*

The following lemma lists some properties of KLM model to be used.

Lemma 5.2. *If $W_c = \langle S_c, \ell_c, \prec_c\rangle$ is a KLM model associated with the preferential relation $|\!\sim$, then*

(1) *$\alpha \;|\!\sim \beta$ iff $\alpha \;|\!\sim_{w_c} \beta$.*

(2) *$\top \leq \alpha$ for any α.*

(3) *If $\langle n, \alpha\rangle$ is a minimal element of $\langle S_c, \prec_c\rangle$, then the valuation n is $\top$-normal.*

(4) *If n is a $\top$-normal valuation, then $\langle n, \top\rangle$ is a minimal element in $\langle S_c, \prec_c\rangle$.*

(5) *If $\langle m, \top\rangle \not\prec_c \langle n, \alpha\rangle$, then $m \models \alpha$.*

(6) *If $\langle n, \alpha\rangle \prec_c \langle m, \beta\rangle$, k is an α-normal valuation and $k \models \beta$, then k is a β-normal valuation.*

(7) *If $\langle m, \top\rangle \not\prec_c \langle n, \alpha\rangle$, then m is an α-normal valuation.*

Proof. (1) See Lemmas 5.17 and 5.16 in [4].

(2) From $\models (\alpha \vee \top) \leftrightarrow \top$ and $\top \;|\!\sim \top$, we have $\alpha \vee \top \;|\!\sim \top$ by the rule **LLE**.

(3) Immediately from (2) and Lemma 5.1.

(4) Otherwise, there exists $\langle m, \alpha\rangle$ such that $\langle m, \alpha\rangle \prec_c \langle n, \top\rangle$, so, by the construction of W_c, we have $m \models false$.

(5) By (2), we have $\top \leq \alpha$, so, if $m \not\models \alpha$, then $\langle m, \top\rangle \prec_c \langle n, \alpha\rangle$. Consequently, $m \models \alpha$.

(6) Immediately from Lemma 5.1 and the construction of the model W_c.

(7) Immediately from Lemma 5.1, (2), (5) and the definition of $\prec_c$. □

In order to establish representation theorem for **P + WRM**, in the following we transform a **KLM** model to a **WRM** model. Suppose that the relation $|\!\sim$ is a preferential inference relation, and the model $W_c = \langle S_c, \ell_c, \prec_c\rangle$ is the KLM model associated with $|\!\sim$. We construct the preferential model $\diamond(W_c) = \langle S, \ell, \prec\rangle$ as follows:

1) $S = S_c$;

2) $\ell = \ell_c$;

3) $\prec = \prec_c \cup \{\langle s, t\rangle : s \in W_t \text{ and } t \notin W_t\}$, where $W_t = \{\langle n, \top\rangle\text{: } \langle n, \top\rangle \in S\}$.

It is obvious that $\diamond(W_c)$ is a **WRM** model. For convenience, in the following, we call the above model $\diamond(W_c)$ the WRM-transform of W_c.

Lemma 5.3. *If $W_c = \langle S_c, \ell_c, \prec_c\rangle$ is a KLM model, then $|\!\sim_{w_c} \subseteq |\!\sim_{\diamond(W_c)}$.*

Proof. Suppose that $\alpha |\!\sim_{w_c} \beta$. We want to show $\alpha |\!\sim_{\diamond(W_c)} \beta$. Let $\langle n, r\rangle$ be minimal in $\langle \| \alpha \|_{\diamond(W_c)}, \prec_{\diamond(W_c)}\rangle$. Since $\| \alpha \|_{\diamond(W_c)} = \| \alpha \|_{W_c}$ and $\prec_c \subseteq \prec_{\diamond(W_c)}$, $\langle n, r\rangle$ is still minimal in $\langle \| \alpha \|_{w_c}, \prec_c\rangle$. By $\alpha |\!\sim_{w_c} \beta$, we have $n \models \beta$. Consequently, $\alpha |\!\sim_{\diamond(W_c)} \beta$. □

It is easy to see that, in general, $|\!\sim_{W_c} \neq |\!\sim_{\diamond(W_c)}$. However, in the following, we will show that, if the relation $|\!\sim_{W_c}$ satisfies the rule **WRM** then $|\!\sim_{W_c} = |\!\sim_{\diamond(W_c)}$ holds.

Lemma 5.4. *The following rule is derivable in* **P + WRM**:

$$\frac{\top |\!\sim \alpha \to \beta, \top \not|\!\sim \neg\alpha}{\alpha |\!\sim \beta}$$

Proof. Suppose that $\top |\!\sim \alpha \to \beta$ and $\top \not|\!\sim \neg\alpha$. Since $\top \not|\!\sim \neg\top$ [6] and the assumptions, by **WRM**, we obtain $\top \wedge \alpha |\!\sim \alpha \to \beta$. By **LLE**, we have $\alpha |\!\sim \alpha \to \beta$. Furthermore, combining it with $\alpha |\!\sim \alpha$, by **AND** and **RW**, we get $\alpha |\!\sim \beta$. □

Lemma 5.5. *If $W_c = \langle S_c, \ell_c, \prec_c\rangle$ is a KLM model and $|\!\sim_{W_c}$ satisfies* **WRM**, *then $|\!\sim_{w_c} \supseteq |\!\sim_{\diamond(W_c)}$.*

Proof. Suppose that $\alpha \not|\!\sim_{w_c} \beta$. We want to show $\alpha \not|\!\sim_{\diamond(W_c)} \beta$. Since $\alpha \not|\!\sim_{w_c} \beta$, there exists a minimal element in $\langle \| \alpha \|_{w_c}, \prec_c\rangle$ satisfying $\neg\beta$. Let $\langle n, q\rangle$ be such state. We consider two cases.

Case 1. Suppose that $\langle n, q\rangle$ is minimal in $\langle \| \alpha \|_{\diamond(W_c)}, \prec_{\diamond(W_c)}\rangle$.

Since $n \models \neg\beta$ and $\langle n, q\rangle$ is a minimal element in $\langle \| \alpha \|_{\diamond(W_c)}, \prec_{\diamond(W_c)}\rangle$, we get $\alpha \not|\!\sim_{\diamond(W_c)} \beta$.

Case 2. Suppose that $\langle n, q\rangle$ is not a minimal in $\langle \| \alpha \|_{\diamond(W_c)}, \prec_{\diamond(W_c)}\rangle$.

Since $\langle n, q\rangle$ is not a minimal element in $\langle \| \alpha \|_{\diamond(W_c)}, \prec_{\diamond(W_c)}\rangle$, there exists $\langle m, \top\rangle$ such that $\langle m, \top\rangle \prec_{\diamond(W_c)} \langle n, q\rangle$ and $m \models \alpha$. So, by Lemma 5.2 (4), we have $\top \not|\!\sim_{w_c} \neg\alpha$. By Lemma 5.2 (1), $\top \not|\!\sim_{w_c} \neg\alpha$ and $\alpha \not|\!\sim_{w_c} \beta$, we obtain $\top \not|\!\sim \neg\alpha$ and $\alpha \not|\!\sim \beta$. Furthermore, by Lemma 5.4, we have $\top \not|\!\sim \alpha \to \beta$. Thus, $\top \not|\!\sim_{w_c} \alpha \to \beta$ and there exists a minimal element $\langle t, \gamma\rangle$ in W_c such that $t \models \alpha$ and $t \models \neg\beta$. By the Lemma 5.2 (3), we have $\langle t, \top\rangle \in S$. It is obvious that $\langle t, \top\rangle$ is minimal in $\langle \| \alpha \|_{\diamond(W_c)}, \prec_{\diamond(W_c)}\rangle$. Consequently, $\alpha \not|\!\sim_{\diamond(W_c)} \beta$. □

By Theorem 3.1, Lemmas 5.2 (1), 5.3 and 5.5, we get the following theorem:

Theorem 5.1. *Suppose that $|\!\sim$ is a preferential inference relation and $W_c = \langle S_c, \ell_c, \prec_c\rangle$ is the KLM model associated with the relation $|\!\sim$. Then the following conditions are equivalent.*

(1) $|\!\sim$ *is a* **P + WRM** *inference relation.*

(2) $|\!\sim_{w_c} = |\!\sim_{\diamond(W_c)}$.

[6] Otherwise, from $\models \neg\top \to \neg\alpha$ and $\top |\!\sim \neg\top$, we have $\top |\!\sim \neg\alpha$ by the rule **RW**, it contradicts the last assumption.

Theorem 5.2. *$|\sim$ is a* $\mathbf{P+WRM}$ *inference relation if and only if there is a* **WRM** *model* $W=\langle S,l,\prec\rangle$ *such that* $|\sim=|\sim_w$.

Proof. Immediately from Lemma 5.2 (1), Theorems 3.1 and 5.1. □

6 Conditional Logic CS4.2

Conditional logics were originally constructed in order to account for properties of conditional statements in natural language. These logics consist of the classical propositional logic augmented with a conditional connective, often written as $\Rightarrow$. This additional connective is necessitated because the material conditional does not adequately reflect linguistic usage of "if-then" constructs. Recently, the use of conditional logics in nonmonotonic reasoning has been explored[13−19]. Among them, Boutilier[13] and Lamarre[17] demonstrate that *preferential* system and *rational* system are equivalent to the 'flat' fragment of conditional logic **C4** and **CT4D** respectively. Their results show clearly that nonmonotonic consequence relation $|\sim$ may be characterized by conditional implication$\Rightarrow$. Furthermore, their researches lay theoretic foundations of transforming the automated reasonings in conditional knowledge base[5] into those in normal modal logics, for which automated theorem proving methods are known.

In the ensuing paragraphs, we will explore the relation between $\mathbf{P+WRM}$ and the conditional logic **CS4.2**. In this section, we introduce the conditional logic **CS4.2**.

The underlying language $L_{\Rightarrow}$ of **CS4.2** is a standard propositional language with a special connective $\Rightarrow$. Well-formed formula of $L_{\Rightarrow}$ is defined as usual. In the following, we denote the set of all well-formed formulae of $L_{\Rightarrow}$ by $Form(L_{\Rightarrow})$.

A *plausibility space*[19] is a pair $(\Omega,\leq)$, where Ω is a set and $\leq$ is a reflexive and transitive relation over Ω. Given two subsets S and T of Ω, we follow the standard technique in [19] of saying that plausibility space $(\Omega,\leq)$ satisfies $S\rightharpoonup T$ if $\forall\ s\in S\ \exists t\in T\cap S$ $(s\leq t$ and $\neg\exists u\in S-T(t\leq u))$.

Lemma 6.1[19]**.**

(1) *If $S\rightharpoonup T_1$ and $S\rightharpoonup T_2$ hold in $(\Omega,\leq)$, then $S\rightharpoonup T_1\cap T_2$ holds in $(\Omega,\leq)$.*

(2) *If $S_1\rightharpoonup T$ and $S_2\rightharpoonup T$ hold in $(\Omega,\leq)$, then $S_1\cup S_2\rightharpoonup T$ holds in $(\Omega,\leq)$.*

(3) *If $S\rightharpoonup T_1$ and $S\rightharpoonup T_2$ hold in $(\Omega,\leq)$, then $S\cap T_1\rightharpoonup T_2$ holds in $(\Omega,\leq)$.*

(4) *If $S\subseteq T$ and $S_1\rightharpoonup S$ holds in $(\Omega,\leq)$, then $S_1\rightharpoonup T$ holds in $(\Omega,\leq)$.*

Definition 6.1. *A partial order R is called weak directed relation if it satisfies the following condition: for any x,y,z, if $R(x,y)$ and $R(x,z)$ then there exists an element t such that $R(y,t)$ and $R(z,t)$.*

Let $M=\langle W,R,V\rangle$ be a Kripke model ⑦, R a weak directed relation over W and $w\in W$, we define the notion $M,w\models\gamma$ as follows:

$M,w\models p$ iff $V(w)\models p$, where p is a primitive proposition;

$M,w\models\neg\alpha$ iff $M,w\not\models\alpha$;

$M,w\models\alpha\wedge\beta$ iff $M,w\models\alpha$ and $M,w\models\beta$;

$M,w\models\alpha\Rightarrow\beta$ iff $S^{\alpha}_{w}\rightharpoonup S^{\beta}_{w}$ holds in $\langle R(w),\ R\Downarrow R(w)\rangle$, where $R(w)=\{t:\langle w,t\rangle\in R\}$, $R\Downarrow R(w)=R\cap(R(w))^2$ and $S^{\alpha}_{w}=\{t\text{: }\langle w,t\rangle\in R$ and $M,t\models\alpha\}$.

The conditional logic of normality (denoted by **CS4.2**), which is complete and sound with respect to the class of weak directed frames, is the smallest $S\subseteq Form(L_{\Rightarrow})$ such that S contains the following axiom schema and is closed under the following rules, where

⑦Introduction about Kripke model is omitted. Detailed introduction concerning modal logic and Kripke model may be found in [20].

$\Box_c\alpha =_{\rm def} \neg\alpha \Rightarrow \alpha$ ⑧⑨,:

(1) All tautology in classical proposition logic.

(2) $\Box_c(A \to B) \to (\Box_c A \to \Box_c B)$.

(3) $\Box_c A \to A$.

(4) $\Box_c A \to \Box_c\Box_c A$.

(5) $\neg\Box_c\neg\Box_c A \to \Box_c\neg\Box_c\neg A$.

(6) $(A \Rightarrow B) \leftrightarrow \Box_c(\Box_c\neg A \vee \neg\Box_c\neg(A \wedge \Box_c(A \to B)))$.

(MP) From $A \to B$ and A infer B.

(NES) From A infer $\neg A \Rightarrow A$.

(US) From A infer A_1, where A_1 is a substitution instance of A.

The following lemma is trivial but useful in the following section.

Lemma 6.2. *If $\langle W, R\rangle$ is a weak directed frame, then* $A \Rightarrow C, \neg(A \wedge B \Rightarrow C), \neg(A \Rightarrow \neg B) \models_{\langle W,R\rangle} \top \Rightarrow \neg A$.

7 The System PWRM*

Nonmonotonic inference relation $|\!\sim$ is a metalingustic object, however, conditional implication $\Rightarrow$ is in object language $L_{\Rightarrow}$, so, in order to explore the connection between them, we follow the method in [13] and interpret $|\!\sim$ as conditional connection in some object language and translate the inference rules of **P + WRM** into the corresponding Hilbert-style axioms. Boutilier[13] extends the language of *conditional* assertions as follows: permitting Boolean combinations of assertions as well as propositional formula (without occurrence of $|\!\sim$) by viewing $|\!\sim$ as a connective. The well-formed formulae of this enriched language are called extended conditional assertions, the set of which is denoted by L_{EC}.

PWRM* is the smallest set $S \subseteq L_{EC}$ containing all tautology in classical proposition logic and the following axioms, and is closed under the following inference rules:

ID $A \,|\!\sim A$,

AND $(A \,|\!\sim B \wedge A \,|\!\sim C) \to A \,|\!\sim B \wedge C$,

OR $(A \,|\!\sim C \wedge B \,|\!\sim C) \to A \vee B \,|\!\sim C$,

CM $(A \,|\!\sim B \wedge A \,|\!\sim C) \to A \wedge B \,|\!\sim C$,

WRM $(A \,|\!\sim C \wedge \neg(A \wedge B \,|\!\sim C) \wedge \neg(A \,|\!\sim \neg B)) \to \top \,|\!\sim \neg A$,

T $(\neg A \,|\!\sim A) \to A$,

LLE from $\models A \leftrightarrow B$ infer $(A \,|\!\sim C) \to (B \,|\!\sim C)$,

RW from $\models A \to B$ infer $(C \,|\!\sim A) \to (C \,|\!\sim B)$,

MP from $A \to B$ and A infer B, and

US from A infer A_1, where A_1 is a substitution instance of A.

Let $M = \langle S, \ell, \prec\rangle$ be a **WRM** model and $s \in S$. The truth of an extended conditional assertion A at s is defined inductively as follows, where $M, s \models_{PWRM^*} A$ means A is true at s.

(1) $M, s \models_{PWRM^*} p$ iff $p \in \ell(s)$, where p is a primitive proposition;

(2) $M, s \models_{PWRM^*} \neg A$ iff $M, s \not\models A$;

(3) $M, s \models_{PWRM^*} A \to B$ iff $M, s \models B$ or $M, s \not\models A$;

(4) $M, s \models_{PWRM^*} A \,|\!\sim B$ iff $A \,|\!\sim_M B$.

Theorem 7.1. $\vdash_{PWRM^*} A$ *iff* $\models_{PWRM^*} A$.

Proof. ($\Rightarrow$) By Theorem 5.2, the proof is trivial.

($\Leftarrow$) It is enough to show that every **PWRM*** consistent set is satisfiable. Let Γ_1 be such a set of formulae, and Γ be any maximal consistent set in L_{EC} containing Γ_1. The

⑧See: Zhu Zhouhui, Two dimensional structure intention theory and nonmonotonic reasoning, Ph.D.Thesis, Nanjing University of Aeronautics and Astronautics, 1998.

⑨It is easy to verfiy that $M, w \models \Box_c\alpha$ iff $\forall t($ if $\langle w, t\rangle \in R_M$ then $M, t \models \alpha)$ where M is a Kripke model.

set $\{A \mid\sim B\colon A \mid\sim B \in \Gamma\} \cup \{A \mid\not\sim B : \neg(A \mid\sim B) \in \Gamma\}$ is denoted by K. It is easy to know that the set K forms a $\mathbf{P}+\mathbf{WRM}$ inference relation. By Theorem 5.2, there exists a **WRM** model $M = \langle S, \ell, \prec\rangle$ such that $A \mid\sim B \in K$ iff $A \mid\sim_M B$. Construct a **WRM** model $M_1 = \langle S_1, \ell_1, \prec_1\rangle$ as follows:

(1) $S_1 = S \cup \{w\}$, where $w \notin S$;

(2) $\prec_1 = \prec \cup \{\langle s, w\rangle : s \in S\}$;

(3) For any $s \in S_1$,

$$\ell_1(s) = \begin{cases} \ell(s), & \text{if } s \neq w; \\ \{p : p \in \Gamma \text{ and } p \text{ is atomic}\}, & \text{otherwise.} \end{cases}$$

Clearly, M_1 is a **WRM** model. In the following, we show $M_1, w \models \Gamma$. We proceed by structural induction on formula to show $M_1, w \models A$ iff $A \in \Gamma$. Obviously, for any atomic variable p, $p \in \Gamma$ iff $M_1, w \models p$. Assume that the property holds for A and B. For the formula with shape $\neg\alpha$ or $\alpha \to \beta$, the proof is trivial. In the following, we deal with formula with shape $\alpha \mid\sim \beta$. Suppose that $A \mid\sim B \in \Gamma$. So, $A \mid\sim_M B$. We consider two cases.

Case 1. Suppose that $\| A \|_M \neq \emptyset$. By the construction of M_1, any minimal point in $\| A \|_{M_1}$ is also minimal in $\| A \|_M$. Hence, $M_1, w \models A \mid\sim B$.

Case 2. Suppose that $\| A \|_M = \emptyset$. Since $\| A \|_M = \emptyset$, we get $A \mid\sim_M \neg A$. Furthermore, we know $A \mid\sim \neg A \in \Gamma$. Since Γ is a maximal $\mathbf{PWRM}^*$ consistent set, by the axiom **T**, we know $\neg A \in \Gamma$ (i.e., $A \notin \Gamma$). By the inductive hypothesis, we gain $M_1, w \models \neg A$. So, $\| A \|_{M_1} = \emptyset$. Hence, $M_1, w \models A \mid\sim B$.

On the other hand, suppose $A \mid\sim B \notin \Gamma$. Since Γ is a maximal $\mathbf{PWRM}^*$ consistent set, we have $\neg(A \mid\sim B) \in \Gamma$. Furthermore, we get $A \mid\not\sim_M B$. Hence, there exists a minimal element (denoted by t) in $\| A \|_M$ such that $M, t \not\models B$. It is obvious that t is still minimal in $\| A \|_{M_1}$, so, $M_1, w \not\models A \mid\sim B$. □

8 PWRM* and CS4.2

In this section, we will explore the relation between $\mathbf{PWRM}^*$ and **CS4.2**. For the sake of convenience, we suppose that the language L is denumerable.

Definition 8.1. $\Lambda =_{\text{def}} \{A : A$ *is a well-formed formula of* $L_{\Rightarrow}$ *without occurrence of nested* $\Rightarrow\}$.

Definition 8.2. *A translation functor* $o : L_{EC} \to \Lambda$ *is defined inductively as follows*:

(1) *If* α *is atomic, then* $\alpha^o = \alpha$;

(2) *If* α *has the form* $\neg\beta$, *then* $\alpha^o = \neg\beta^o$;

(3) *If* α *has the form* $\beta \to \gamma$, *then* $\alpha^o = \beta^o \to \gamma^o$;

(4) *If* α *has the form* $\beta \mid\sim \gamma$, *then* $\alpha^o = \beta^o \Rightarrow \gamma^o$.

Definition 8.3. *Let* Γ *be a maximal* $\mathbf{PWRM}^*$ *consistent set,* Γ *is called* I*-type maximal consistent set if* Γ *satisfies the following condition*:

$$\top \mid\sim \alpha \in \Gamma \text{ or } \top \mid\sim \neg\alpha \in \Gamma, \text{ for any } \alpha \in \mathit{Form}(L).$$

Otherwise, Γ *is called* II*-type maximal consistent set.*

Lemma 8.1. *If* Γ *is a maximal* $\mathbf{PWRM}^*$ *consistent set, then* $S_\Gamma = \{\alpha : \top \mid\sim \alpha \in \Gamma\}$ *is consistent and closed in classical propositional logic.*

Proof. Suppose that S_Γ is inconsistent. Thus, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha \in S_\Gamma$ such that $\vdash \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \to \neg\alpha$. From $\top \mid\sim \alpha_i \in \Gamma$ $(1 \leq i \leq n)$, we have $\top \mid\sim \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \in \Gamma$. Furthermore, by **RW** and **MP,** we get $\top \mid\sim \neg\alpha \in \Gamma$. Since Γ is a maximal consistent set, by **AND**, we have $\top \mid\sim \neg\top \in \Gamma$. By **T** and **MP**, we obtain $\neg\top \in \Gamma$. This contradicts that Γ is a consistent set. Analogously, we may show that S_Γ is closed. □

Definition 8.4. *Let Γ be a maximal* **PWRM*** *consistent set and* $\neg(\alpha \mid\!\sim \beta) \in \Gamma$. *The set* $\Gamma_{\alpha|\not\sim\beta}$ *is defined as follows*:

$$\Gamma_{\alpha|\not\sim\beta} = \begin{cases} S_\Gamma \cup \{\neg\beta\}, & \text{if } \alpha \in S_\Gamma; \\ S_\Gamma, & \text{otherwise.} \end{cases}$$

Lemma 8.2. *Let Γ be a maximal* **PWRM*** *consistent set and* $\neg(\alpha \mid\!\sim \beta) \in \Gamma$, *then* $\Gamma_{\alpha|\not\sim\beta}$ *is consistent.*

Proof. By Lemma 8.1, it is enough to show that $\Gamma_{\alpha|\not\sim\beta}$ is consistent when $\alpha \in S_\Gamma$. Let $\neg(\alpha \mid\!\sim \beta) \in \Gamma$ and $\alpha \in S_\Gamma$. Suppose that $\Gamma_{\alpha|\not\sim\beta}$ is inconsistent. Thus, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in S_\Gamma$ such that $\vdash \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n \to \beta$. By Lemma 8.1, we have $\beta \in S_\Gamma$. Consequently, $\top \mid\!\sim \beta \in \Gamma$. From $\top \mid\!\sim \alpha \in \Gamma$, $\top \mid\!\sim \beta \in \Gamma$ and $\neg(\top \mid\!\sim \neg\top) \in \Gamma$, by **WRM**, **LLE**, **MP** and the fact that Γ is a maximal consistent set, we gain $\alpha \mid\!\sim \beta \in \Gamma$. It contradicts $\neg(\alpha \mid\!\sim \beta) \in \Gamma$. □

Lemma 8.3. If Γ *is a maximal consistent set and* $S_\Gamma \not\vdash \alpha \to \beta$, *then* $\neg(\alpha \mid\!\sim \beta) \in \Gamma$.

Proof. Suppose that $\neg(\alpha \mid\!\sim \beta) \notin \Gamma$. Since Γ is a maximal consistent set, we have $\alpha \mid\!\sim \beta \in \Gamma$. Furthermore, by **RW** and **MP**, we gain $\alpha \mid\!\sim \alpha \to \beta \in \Gamma$. On the other hand, by **ID**, **RW** and **MP**, we obtain $\neg\alpha \mid\!\sim \alpha \to \beta \in \Gamma$. Further, by **OR**, **LLE** and **MP**, we get $\top \mid\!\sim \alpha \to \beta \in \Gamma$. Hence, $\alpha \to \beta \in S_\Gamma$. This contradicts $S_\Gamma \not\vdash \alpha \to \beta$. □

Lemma 8.4. *If Γ is a* II*-type maximal consistent set and* $S_\Gamma \not\vdash \alpha \to \beta$, *then there exists* $\Gamma_{\gamma|\not\sim\delta}$ *which is consistent with* $\neg(\alpha \to \beta)$.

Proof. Suppose that there does not exist such $\Gamma_{\gamma|\not\sim\delta}$. Thus, for any $\neg(\top \mid\!\sim \gamma) \in \Gamma$, we have $S_\Gamma \cup \{\neg\gamma\} \vdash \alpha \to \beta$ or $S_\Gamma \vdash \alpha \to \beta$. Since $S_\Gamma \not\vdash \alpha \to \beta$, we have $S_\Gamma \cup \{\neg\gamma\} \vdash \alpha \to \beta$. Moreover, since Γ is a II-type maximal consistent set, there exists δ such that $\neg(\top \mid\!\sim \delta) \in \Gamma$ and $\neg(\top \mid\!\sim \neg\delta) \in \Gamma$. Hence, $S_\Gamma \cup \{\neg\delta\} \vdash \alpha \to \beta$ and $S_\Gamma \cup \{\delta\} \vdash \alpha \to \beta$. Consequently, $S_\Gamma \vdash \alpha \to \beta$. This contradicts $S_\Gamma \not\vdash \alpha \to \beta$. □

Definition 8.5. *Suppose that Γ is a* II*-type maximal consistent set. If* $S_\Gamma \not\vdash \alpha \to \beta$ *and* $\Gamma_{\gamma|\not\sim\delta} \cup \{\neg(\alpha \to \beta)\}$ *is consistent, then we define* $\Gamma^{\alpha|\not\sim\beta}_{\gamma|\not\sim\delta}$ *as follows*:

$$\Gamma^{\alpha|\not\sim\beta}_{\gamma|\not\sim\delta} =_{\text{def}} \Gamma_{\gamma|\not\sim\delta} \cup \{\neg(\alpha \to \beta)\}.$$

Definition 8.6. *Let Γ be a maximal consistent set,* Ω_Γ *is defined as follows*:

$$\Omega_\Gamma = \begin{cases} \{\Gamma_{\gamma|\not\sim\delta} : \neg(\gamma \mid\!\sim \delta) \in \Gamma\}, & \textit{if } \Gamma \textit{ is a } \text{I}\textit{-type consistent set}; \\ \Psi_\Gamma, & \textit{otherwise}, \end{cases}$$

where $\Psi_\Gamma = \{\Gamma_{\gamma|\not\sim\delta} : \neg(\gamma \mid\!\sim \delta) \in \Gamma\} \cup \{\Gamma^{\alpha|\not\sim\beta}_{\gamma|\not\sim\delta} : S_\Gamma \not\vdash \alpha \to \beta, \Gamma_{\gamma|\not\sim\delta} \cup \{\neg(\alpha \to \beta)\}$ *is consistent and* $\neg(\gamma \mid\!\sim \delta) \in \Gamma\}$.

Theorem 8.1. *If Γ is a maximal* **PWRM*** *consistent set, then there exists a* **CS4.2** *model, which satisfies the set* $\Gamma^o = \{\alpha^o : \alpha \in \Gamma\}$.

Proof. Let $\Gamma^+ = \{\alpha \mid\!\sim \beta : \alpha \mid\!\sim \beta \in \Gamma\}$ and $\Gamma^- = \{\alpha \mid\!\not\sim \beta : \neg(\alpha \mid\!\sim \beta) \in \Gamma\}$. Thus, $\Gamma^+ \cup \Gamma^-$ is a **P + WRM** inference relation, and therefore there exists a **WRM** model $\langle S, \prec, \ell\rangle$ generating $\Gamma^+ \cup \Gamma^-$. According to the proof of Theorem 7.1, there exists a **WRM** model $M_1 = \langle S \cup \{w_0\}, \prec_1, \ell_1\rangle$ such that $M_1, w_0 \models \Gamma$ and $s \prec_1 w_0$ for any $s \in S$. In the following, we will construct a **CS4.2** model M and show $M, w_0 \models \Gamma^o$.

Since L is denumerable, Γ^- is denumerable. Hence, there exists an infinite sequence $s_0, s_1, s_2, \ldots, s_i \ldots$ $(i \geq 0)$ satisfying the following conditions:

(1) $s_0 = S_\Gamma$,

(2) $s_i \in \Omega_\Gamma$ $(i \geq 1)$, and

(3) for any natural number n and $x \in \Omega_\Gamma$, there exists a natural number m such that $m > n$ and $s_m = x$.

Construct a **CS4.2** model $M = \langle D_M, \ell_M, \prec_M\rangle$ as follows:

(1) $D_M = S \cup \{w_0\} \cup \{s_i : i \geq 0\}$,

(2) $\prec_M = (\prec_1)^{-1} \cup \{\langle s_i, s_j\rangle : i \leq j\} \cup \{\langle w, s_i\rangle : w \in S \cup \{w_0\}$ and i is a natural number$\}$ $\cup \{\langle x, x\rangle : x \in D_M\}$,

(3) By Lemmas 8.2 and 8.4, for any $x \in \Omega_\Gamma$, x is consistent. Now, for any s_i, we choice arbitrarily a word which satisfies s_i and denote it by $choice(s_i)$. Construct ℓ_M as follows:

$$\text{for any } s \in D_M,\ \ell_M(s) = \begin{cases} \ell_M(s) = \ell_1(s), & \text{if } s \in S \cup \{w_0\}; \\ \ell_M(s) = choice(s), & \text{if } s \in \{s_i : i \geq 0\}. \end{cases}$$

It is obvious that the relation $\prec_M$ is weak directed. In the following, we will show that $M, w_0 \models \Gamma^o$. We proceed by structural induction on formula.

(1) Obviously, if $\alpha \in \Gamma^o$ and $\alpha \in Form(L)$, then $M, w_0 \models \alpha$.

(2) Suppose that $\alpha \Rightarrow \beta \in \Gamma^o$. Thus, $\alpha \mid\!\sim \beta \in \Gamma$. If $\parallel \alpha \parallel_M = \emptyset$, then $M, w_0 \models \alpha \Rightarrow \beta$ holds trivially. If $\parallel \alpha \parallel_M \neq \emptyset$, let w be an arbitrary element in $\parallel \alpha \parallel_M$. We consider two cases.

Case 2.1. Suppose that $w \in \{s_0, s_1, \ldots, s_i, \ldots\}$.

Since $\alpha \mid\!\sim \beta \in \Gamma$ and Γ is a maximal consistent set, we get $\top \mid\!\sim \alpha \to \beta \in \Gamma$[10]. Hence, $M, s_i \models \alpha \to \beta$ for any i. Therefore, $M, w \models \alpha \wedge \beta$, and for any s, if $w \prec s$ then $M, s \models \alpha \to \beta$.

Case 2.2. Suppose that $w \in S \cup \{w_0\}$.

By the smoothness of $\parallel \alpha \parallel_{M_1}$, w is minimal in $(S \cup \{w_0\}) \cap \parallel \alpha \parallel_{M_1}$ or there exists a minimal element (denoted by t) such that $t \prec_1 w$. In both cases, we can conclude that there exists t_1 ($t_1 = w$ or $t_1 \prec_1 w$) such that $M_1, t_1 \models \alpha \wedge \beta$ and for any u, if $u \prec_1 t_1$ then $M_1, u \models \alpha \to \beta$. Furthermore, together with the construction of M, we know that there exists t_1 satisfying the following conditions:

(a) $w \prec_M t_1$,

(b) $M, t_1 \models \alpha \wedge \beta$, and

(c) for any u, if $t_1 \prec_M u$ then $M, u \models \alpha \to \beta$.

With both Case 2.1 and Case 2.2, we get $M, w_0 \models \alpha \Rightarrow \beta$.

(3) Suppose that $\neg(\alpha \Rightarrow \beta) \in \Gamma^o$. Thus, $\neg(\alpha \mid\!\sim \beta) \in \Gamma$. We consider two cases.

Case 3.1. Suppose that Γ is a II-*type* consistent set.

Case 3.1.1. Suppose that $S_\Gamma \vdash \alpha \to \beta$.

Hence, $\top \mid\!\sim \alpha \to \beta \in \Gamma$ and $\top \mid\!\sim \neg\alpha \in \Gamma$[11]. Consequently, we obtain $\neg\alpha \in S_\Gamma$. On the other hand, since $\neg(\alpha \mid\!\sim \beta) \in \Gamma$, we get $\alpha \mid\!\not\sim_{M_1} \beta$. Hence, there exists a minimal element s in $\parallel \alpha \parallel_{M_1}$ such that $\ell_1(s) \models \alpha \wedge \neg\beta$. So, by $\neg\alpha \in S_\Gamma$ and the construction of M, for any $x \in D_M$, we have $\ell_M(x) \not\models \alpha$ if $s \prec x$. Consequently, $M, w_0 \models \neg(\alpha \Rightarrow \beta)$.

Case 3.1.2. Suppose that $S_\Gamma \not\vdash \alpha \to \beta$.

By Lemma 8.4 and the construction of M, it is easy to verify that $M, w_0 \models \Box_c \Diamond_c \neg(\alpha \to \beta)$. Therefore, $M, w_0 \models \neg(\alpha \Rightarrow \beta)$.

Case 3.2. Suppose that Γ is a I-type consistent set.

Since Γ is a I-type consistent set, for any $\alpha \in Form(L)$, we get $\alpha \in S_\Gamma$ or $\neg\alpha \in S_\Gamma$. If $\neg(\alpha \to \beta) \in S_\Gamma$, then we get $M, w_0 \models \Box_c \Diamond_c \neg(\alpha \to \beta)$. If $\alpha \to \beta \in S_\Gamma$, then the proof is analogous with Case 3.1.1. Therefore, $M, w_0 \models \neg(\alpha \Rightarrow \beta)$.

For any formula $\alpha \in \Gamma^o$, since α possesses one of the above three forms or may be translated equivalently into the form $(\alpha_{11} \vee \alpha_{12} \vee \cdots \vee \alpha_{1m}) \wedge (\alpha_{21} \vee \alpha_{22} \vee \cdots \vee \alpha_{2m}) \wedge \cdots \wedge (\alpha_{n1} \vee \alpha_{n2} \vee \cdots \vee \alpha_{nm})$, in which α_{ij} possesses one of the above three forms. Hence, if $\alpha \in \Gamma^o$ then $M, w_0 \models \alpha$ by the proof of (1)–(3). □

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[10] See the proof of Lemma 8.3.

[11] Otherwise, since $T \mid\!\sim \alpha \to \beta \in \Gamma$, $\neg(T \mid\!\sim \neg\alpha) \in \Gamma$, and $\neg(T \mid\!\sim \neg T) \in \Gamma$, by **WRM**, **LLE** and **MP**, we get $\alpha \mid\!\sim \alpha \to \beta \in \Gamma$. Furthermore, since Γ is a maximal consistent set, by **ID**, **AND** and **MP**, we have $\alpha \mid\!\sim \beta \in \Gamma$. This contradicts $\neg(\alpha \mid\!\sim \beta) \in \Gamma$.

Theorem 8.2. *For any $\alpha \in L_{EC}$, $\vdash_{PWRM^*} \alpha$ iff $\vdash_{\mathrm{CS4.2}} \alpha^o$.*

Proof. ($\Leftarrow$) Suppose that $\nvdash_{PWRM^*} \alpha$ and $\vdash_{\mathrm{CS4.2}} \alpha^o$. Thus, there exists a maximal **PWRM*** consistent set S such that $\neg\alpha \in S$. By Theorem 8.1, we know that there is a **CS4.2** model which satisfies $\neg\alpha^o$. This contradicts the soundness of **CS4.2**.

($\Rightarrow$) By Lemmas 6.1 and 6.2, it is easy to verify that the translations of all axioms and rules in **PWRM*** by the functor o is valid in **CS4.2** model. Thus, for any $\alpha \in L_{EC}$, if $\vdash_{PWRM^*} \alpha$ then $\models_{\mathrm{CS4.2}} \alpha^o$. Furthermore, by the completeness of **CS4.2**, we obtain $\vdash_{\mathrm{CS4.2}} \alpha^o$. □

Definition 8.7. *For any $\alpha \in \Lambda$, the translation of α into L_{EC} (denoted by α^*) is defined inductively as follows*:

(1) *If α is atomic, then $\alpha^* = \alpha$;*

(2) *If α has the form $\neg\beta$, then $\alpha^* = \neg\beta^*$;*

(3) *If α has the form $\beta \to \gamma$, then $\alpha^* = \beta^* \to \gamma^*$;*

(4) *If α has the form $\beta \Rightarrow \gamma$, then $\alpha^* = \beta^* \mid\!\sim \gamma^*$.* □

Theorem 8.3. *For any $\alpha \in \Lambda$, $\vdash_{PWRM^*} \alpha^*$ iff $\vdash_{\mathrm{CS4.2}} \alpha$.*

Proof. By Theorem 8.2, it is trivial. □

The above Theorems 8.2 and 8.3 demonstrate that **P + WRM** is equivalent to 'flat' fragment of conditional logic **CS4.2**, where 'flat' fragment means tautology in CS4.2 without occurrence of nested $\Rightarrow$.

9 Quasi-Linear Model: A Special Form of PRC Model

Bezzazi, Makinson and Pérez established the representation theorem for **P + RT** in terms of quasi-linear model. However, they left open the question of representation theorems for the weaker postulate sets **P + RC** and **P + WD**[8]. In [9], we introduce valuation-ranked model and PRC model, and solve those open questions. In this section, we will explore the relation between PRC model and quasi-linear model. Main result reveals that quasi-linear model is a special form of PRC model.

Definition 9.1[9]**.** *A preferential model $W = \langle S, l, \prec\rangle$ is said to be PRC model iff it satisfies the following conditions*:

(1) *$W = \langle S, l, \prec\rangle$ is valuation-ranked,*

(2) *for any $v \in l(\min(S))$ and $s \in S$, if $l(s) \notin l(\min(S))$ and $l^{-1}(l(s))$ is not the minimum element of the linear order $\sqsubset\Downarrow \{l^{-1}(v) : v \in rang(l) - l(\min(S))\}$, then there exists $t \in S$ such that $t \prec s$ and $l(t) = v$, and*

(3) *if $l^{-1}(v_0)$ is the minimum element of the linear order $\sqsubset\Downarrow \{l^{-1}(v) : v \in rang(l) - l(\min(S))\}$, then, for any $v \in l(\min(S))$ such that $\exists t\ (t \in l^{-1}(v_0)$ and $v \notin l(\{w : w \prec t\}))$, there exists $s \in l^{-1}(v_0)$ such that $l(\{t : t \prec s\}) = \{v\}$.*

In [9], we have established the following theorem in the framework of finite language.

Theorem 9.1[9]**.** *$\mid\!\sim$ is a* **P + RC** *inference relation if and only if there exists a PRC model $W = \langle S, l, \prec\rangle$ such that $\mid\!\sim = \mid\!\sim_w$.*

Definition 9.2[8]**.** *A model $W = \langle S, l, \prec\rangle$ is said to be parsimonious iff for every state $s \in S$ there is a formula α such that $s \in \min(\| \alpha \|)$.*

Lemma 9.1. *If $W = \langle S, l, \prec\rangle$ is an injective quasi-linear model, then W is a valuation-ranked preferential model.*

Proof. Since W is a quasi-linear model, $\prec\Downarrow (S - \min(S))$ is a strict linear order. By injectivity, $l(S - \min(S)) \cap l(\min(S)) = \emptyset$ and $l^{-1}(v)$ is a single set for any $v \in rang(l)$. Thus, the relation $\sqsubset\Downarrow \{l^{-1}(v) : v \in rang(l) - l(\min(S))\}$ is strict linear. Hence, W is a *valuation-ranked* preferential model. □

Lemma 9.2. *If $W = \langle S, l, \prec\rangle$ is a quasi-linear model, then $s \prec t$ for any $s \in \min(S)$ and $t \in S - \min(S)$.*

Proof. This follows immediately from the fact that W is a ranked model. □

Lemma 9.3. *If $W = \langle S, l, \prec \rangle$ is an injective quasi-linear model, then W is an injective PRC model.*

Proof. By Lemma 9.1, W is an injective *valuation-ranked* model. Suppose that $v \in l(\min(S))$, $s \in S$ and $l(s) \notin l(\min(S))$. Thus, there exists $t \in \min(S)$ such that $l(t) = v$. Obviously, $s \in S - \min(S)$. Furthermore, by Lemma 9.2, we get $t \prec s$. So, W satisfies the conditions (2) and (3) in the definition of PRC model. Hence, W is an injective PRC model. □

Lemma 9.4. *If $W = \langle S, l, \prec \rangle$ is a quasi-linear model, then there exists a quasi-linear model $W_1 = \langle S_1, l_1, \prec_1 \rangle$ satisfying the following conditions*:

(1) $\forall s, t \in \min(S_1)(l_1(s) = l_1(t) \Rightarrow s = t)$, *and*

(2) $|\!\sim_W = |\!\sim_{W_1}$.

Proof. We define the binary relation $\cong$ over $\min(S)$ as follows:

$$s \cong t \text{ iff } l(s) = l(t), \text{ for any } s, t \in \min(S).$$

It is easy to show that $\cong$ is an equivalence relation. We construct $W_1 = \langle S_1, l_1, \prec_1 \rangle$ as follows:

(a) $S_1 = (S - \min(S)) \cup \{[t] : t \in \min(S)\}$, where $[t] = \{s : t \cong s \text{ and } s \in \min(S)\}$,

(b) for any $s \in S_1$, $l_1(s) = \begin{cases} l(s), & \text{if } s \in S - \min(S) \\ l(t), & \text{if there exists } t \in \min(S) \text{ such that } s = [t] \end{cases}$, and

(c) $\prec_1 = \prec \Downarrow (S - \min(S)) \cup \{\langle [t], s \rangle : s \in S - \min(S) \text{ and } t \in \min(S)\}$.

By Lemma 9.2 and the construction of $\cong$, it is easy to know that W_1 satisfies the condition (1), and the above construction is well-defined, i.e., this definition does not depend on the choice of the representative of $[t]$. Furthermore, by the construction of W_1, we get $l(\min(\|\alpha\|_W)) = l_1(\min(\|\alpha\|_{W_1}))$ for any $\alpha \in Form(L)$. Hence, $|\!\sim_W = |\!\sim_{W_1}$. □

Theorem 9.2. *If $W = \langle S, l, \prec \rangle$ is a quasi-linear model, then there exists a quasi-linear model $W_1 = \langle S_1, l_1, \prec_1 \rangle$ satisfying the following conditions*:

(1) W_1 *is injective,*

(2) W_1 *is parsimonious, and*

(3) $|\!\sim_W = |\!\sim_{W_1}$.

Proof. Without loss of generality, by Lemma 9.4, we may suppose that W satisfies the following condition:

$$\text{if } l(s) = l(t) \text{ then } s = t, \text{ for any } s, t \in \min(S).$$

Construct $W_1 = \langle S_1, l_1, \prec_1 \rangle$ as follows:

(a) $S_1 = \cup\{\Delta(\alpha) : \alpha \in Form(L)\}$, where $\Delta(\alpha) = \min(\{s : s \in S \text{ and } l(s) \models \alpha\})$,

(b) $l_1 = l \Downarrow S_1$, and

(c) $\prec_1 = \prec \Downarrow S_1$.

It is obvious that W_1 is a preferential model. Since W is a quasi-linear model, by the above construction, $\prec_1$ satisfies the following conditions:

1) $\forall x, y, z \in S_1($ if $x \not\prec_1 y$, $y \not\prec_1 x$ and $z \prec_1 x$ then $z \prec_1 y)$;

2) $\forall x, y, z \in S_1($ if $x \prec_1 y$ and $x \prec_1 z$ then $z \prec_1 y$ or $y \prec_1 z$ or $y = z)$.

Hence, W_1 is a quasi-linear preferential model. Furthermore, by the construction of S_1, we have $w \in \min(\| \alpha \|_{W_1})$ for any $\alpha \in Form(L)$ and $w \in \triangle(\alpha)$. So, W_1 is parsimonious. In the following, we verify that W_1 is injective. Suppose that there exist $s, t \in S_1$ such that $s \neq t$ and $l_1(s) = l_1(t)$. By the construction of S_1, we have $s \not\prec t$ and $t \not\prec s$. Since W_1 is a quasi-linear model, by Lemma 9.2, $s, t \in \min(S_1)$. It is easy to show that $\min(S) = \min(S_1)$. Hence, $s, t \in \min(S)$. This contradicts the assumption that, for any $s, t \in \min(S)$, if $l(s) = l(t)$ then $s = t$. Thus, W_1 is injective. Furthermore, by the construction of W_1, we get $l(\min(\| \alpha \|_W)) = l_1(\min(\| \alpha \|_{W_1}))$ for any $\alpha \in Form(L)$. Hence, $|\!\sim_W = |\!\sim_{W_1}$. □

Immediately from Theorem 9.2 and Lemma 9.3, we get the following theorem.

Theorem 9.3. *If $W = \langle S, l, \prec \rangle$ is a quasi-linear model, then there exists an injective PRC model W_1 such that $|\!\sim_W = |\!\sim_{W_1}$.*

The above theorem reveals that, for each quasi-linear model W, there exists an injective PRC model which is equivalent to W in the sense to generate the same nonmonotonic inference relation. Bezzazi, Makinson and Pérez establish a representation theorem for $\mathbf{P} + \mathbf{RT}$ in terms of quasi-linear model (see Theorem 2.2) in [8]. By the above result, we also establish a representation theorem for $\mathbf{P} + \mathbf{RT}$ in terms of injective PRC model as follows.

Theorem 9.4. $|\!\sim$ *is a* $\mathbf{P} + \mathbf{RT}$ *inference relation if and only if there exists an injective PRC model $W = \langle S, l, \prec \rangle$ such that $|\!\sim = |\!\sim_W$.*

10 PRC Models and Injective-Closed

Definition 10.1. *Let S be a set of preferential models. We say that S is injective-closed if for any $W \in S$, there exists an injective model $W_1 \in S$ such that $|\!\sim_W = |\!\sim_{W_1}$.*

A number of results in literatures reveal that some preferential model sets are injective-closed, e.g., filter model sets, quasi-linear model set, ranked model set and so on[6,8]. In this section, we will show that the set of all PRC models is not injective-closed.

Lemma 10.1. *Suppose that $W = \langle S, l, \prec \rangle$ is an injective PRC model. If $s \in \min(S)$ and $t \in S - \min(S)$, then $s \prec t$.*

Proof. Suppose that there exists $s \in \min(S)$ and $t \in S - \min(S)$ such that $s \not\prec t$. Since l is injective and $t \in S - \min(S)$, we have $l(t) \notin l(\min(S))$. We consider two cases.

Case 1. Suppose that $\{t\}$ is not the minimum element of the linear order $\sqsubset\Downarrow \{l^{-1}(v) : v \in rang(l) - l(\min(S))\}$ ⑫. Since l is injective, by the condition (2) in the definition of PRC model, we have $s \prec t$. This contradicts $s \not\prec t$.

Caee 2. Suppose that $\{t\}$ is the minimum element of the linear order $\sqsubset\Downarrow \{l^{-1}(v) : v \in rang(l) - l(\min(S))\}$. Since l is injective and $s \not\prec t$, we get $l(s) \notin l(\{u : u \prec t\})$. On the other hand, since l is injective, by the condition (3) in the definition of PRC model, we get $l(\{u : u \prec t\}) = \{l(s)\}$, this contradicts $l(s) \notin l(\{u : u \prec t\})$.

From the above two cases, we get $s \prec t$ for any $s \in \min(S)$ and $t \in S - \min(S)$. □

Lemma 10.2. *If $W = \langle S, l, \prec \rangle$ is an injective PRC model, then W is a ranked model.*

Proof. Construct a totally strict order set $\langle \Phi, \rhd \rangle$ and function $f : S \to \Phi$ as follows:

(1) $\Phi = (S - \min(S)) \cup \{s_0\}$, where $s_0 \notin S$;

(2) $\rhd = \prec\Downarrow (S - \min(S)) \cup \{\langle s_0, t \rangle : t \in S - \min(S)\}$; and

(3) for each $s \in S$, $f(s) = \begin{cases} s, & \text{if } s \in S - \min(S); \\ s_0, & \text{if } s \in \min(S). \end{cases}$

Since W is an injective PRC model, $\prec\Downarrow (S - \min(S))$ is a strict linear order. Furthermore, by Lemma 10.1, $\rhd$ is a strict linear order, and $s \prec t$ iff $f(s) \rhd f(t)$ for any s, $t \in S$. Thus, W is a *ranked* model. □

Lemma 10.3. *If $W = \langle S, l, \prec \rangle$ is an injective PRC model, then W is an injective quasi-linear model.*

Proof. By Lemma 10.2, W is an injective ranked model. Since $\sqsubset\Downarrow \{l^{-1}(v) : v \in rang(l) - l(\min(S))\}$ is a strict linear order, by the definition of $\sqsubset$ and injectivity, we get $\prec\Downarrow \{s : l(s) \in rang(l) - l(\min(S))\}$ is a strict linear order. By injectivity, $l(S - \min(S)) \cap l(\min(S)) = \emptyset$. Thus, $\prec\Downarrow (S - \min(S))$ is a strict linear order. Suppose that $r \prec s$ and $r \prec t$. We have $s, t \in S - \min(S)$. So, $s = t$ or $s \prec t$ or $t \prec s$. Hence, W is an injective quasi-linear model. □

By Theorems 9.4, 2.2 and 9.2, Proposition 4.1, and Lemma 10.3, we get the following theorem:

⑫Notice: Since l is injective, $l^{-1}(v)$ is a single set for each $v \in rang(l)$.

Theorem 10.1. *The set of all PRC models is not injective-closed.*

11 Conclusion and Further Work

We introduce a non-Horn rules **WRM** that is a weak form of *rational monotony*, and explore the effects of adding this non-Horn rule to the rules for preferential inference. We establish the representation theorem for $\mathbf{P}+\mathbf{WRM}$, compare the strength of **WRM** with some non-Horn rules appeared in literatures, and demonstrate that $\mathbf{P}+\mathbf{WRM}$ is equivalent to 'flat' fragment of conditional logic **CS4.2**.

Incidentally, WRM-transform may be regarded as a transformation of preferential models, this view reflects the idea that we may obtain desired models through transforming existent models (e.g., KLM models). This standpoint differs from the approaches in [4–6, 8, 11]. They all pay attention to defining a preferential model directly based on a given inference relation.

In this paper, we also explore the relation between PRC model and quasi-linear model. Main result reveals that quasi-linear model is a special form of PRC model, in other words, for any quasi-linear model W, there exists an injective PRC model which generates the same nonmonotonic inference relation with W. On the other hand, we show that the set of all PRC models is not injective-closed.

From the results in this paper and [8], we know that some preferential model sets are injective-closed (e.g., the set of quasi-linear models and the set of ranked models) and some not (e.g., the set of all preferential models and the set of PRC models). By the results in [6] and [11], it is easy to know that when the language is finite, given a preferential inference relation $|\!\sim$, if $|\!\sim$ satisfies the following property **INJ** presented in [6], then the set of its preferential models is injective-closed.

INJ $C(\alpha \vee \beta) \subseteq Cn(C(\alpha) \cup C(\beta))$, where Cn is classical Tarski operation and $C(\alpha) = \{\beta : \alpha \,|\!\sim \beta\}$.

However, when the language is infinite, the results in [6] and [11] could not imply the above conclusion. This would be a good topic for further researches.

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