On natural deduction system for nonmonotonic reasoning

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Area of Topic: Knowledge representation, nonmonotonic reasoning, belief revision

ABSTRACT

This paper is concerned with the problem of how to construct a natural deduction system for nonmonotonic reasoning. Three possible constructions are investigated, the ones which consists of (1) pure nonmonotonic inference rules; (2) nonmonotonic inference rules plus a nonmonotonic tautology base; or (3) nonmonotonic inference rules plus a nonmonotonic conditional assertion base. Based on a framework of nonmonotonic reasoning, called $\mathbf{RN}$ frame, we show that what can be entailed from a given set of premises in the first kind of systems are exactly those can be done in the classical logic. The second sort of systems can entail more only when the premises of an inference relation is consistent with background knowledge, which is generally viewed as a trivial case in nonmonotonic reasoning. A system constructed in the third way could differ from classical logic in non-trivial case, but it is still reducible to a system with the classical derivability and an ordering of knowledge base.

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Abstract

This paper is concerned with the problem of how to construct a natural deduction system for nonmonotonic reasoning. Three possible constructions are investigated, the ones which consists of (1) pure nonmonotonic inference rules; (2) nonmonotonic inference rules plus a nonmonotonic tautology base; or (3) nonmonotonic inference rules plus a nonmonotonic conditional assertion base. Based on a framework of nonmonotonic reasoning, called $\mathbf{RN}$ *frame, we show that what can be entailed from a given set of premises in the first kind of systems are exactly those can be done in the classical logic. The second sort of systems can entail more only when the premises of an inference relation is consistent with background knowledge, which is generally viewed as a trivial case in nonmonotonic reasoning. A system constructed in the third way could differ from classical logic in non-trivial case, but it is still reducible to a system with the classical derivability and an ordering of knowledge base.*

Keywords: *nonmonotonic reasoning, belief revision, natural deduction, knowledge representation.*

1 Introduction

A deduction system for the classical propositional or first-order logic consists of a set of axioms and a set of inference rules. For any given set of premises Γ and any formula A, A is logical consequence of Γ if and only if A can be derived from Γ by using the inference rules and axioms. The main relationship between formulas in such a system is the *inference relation*,or *derivability*, denoted by $\vdash$. Varying with differences in axioms and/or inference rules, there are a variety of such systems available for classical propositional or first-order logic.

A deductive system using only rules is generally called a *natural deduction system*. The well-known natural deductive systems for the classical propositional logic are Gentzen's system(see [13]) and Kleene's system ([12]). Both of them can be reduced to a **Natural deduction system** which consists of only the following four inference rules:

1 *Reflexivity:*
If $A \in \Gamma$,*then* $\Gamma \vdash A$.

2 *Reductio ad Absurdum:*
If $\Gamma, A \vdash B, \neg B$, *then* $\Gamma \vdash \neg A$.

3 *Implication/Deduction Theorem:*
If $\Gamma, A \vdash B$, *then* $\Gamma \vdash A \to B$.

4 *Modus Ponens:*
If $\Gamma \vdash A \to B$ *and* $\Gamma \vdash A$, *then* $\Gamma \vdash B$.

A natural question is: *Can we have a natural deduction system for nonmonotonic reasoning?*
It is obvious that not all the inference rules of classical logic could be kept in the intended nonmonotonic deductive system. Therefore it is interesting to know that what kinds of inference rules could be kept in the nonmonotonic deductive system, or more generally, what inference rules specify nonmonotonic reasoning? An early attempt to this end is given by Gabbay [6]. Gabbay argued that the following three inference rules are basic for nonmonotonic inference relation "$\mid\!\sim$":

1 Reflexivity:
If $A \in \Gamma$, $\Gamma \mid\!\sim A$.

2 Cut:
If $\Gamma \mid\!\sim A$ *and* $\Gamma, A \mid\!\sim B$, *then* $\Gamma \mid\!\sim B$.

3 Restricted Monotonicity:
If $\Gamma \mid\!\sim A$ *and* $\Gamma \mid\!\sim B$, *then* $\Gamma, A \mid\!\sim B$.

Many other inference rules for nonmonotonic reasoning were put forward ([16] [17] [3][14][15][10][11][22][2]etc). Most of the rules can be expressed in two versions: *finite* or *infinite*, differentiating finite and infinite premises of an inference relation. Some of rules have only one version, such as *Supracompactness*, which only makes sense in the infinite case. The expression of an inference rule also has two flavors:

- *Gentzen style* (using nonmonotonic consequence relation $\mid\sim$) and

- *Tarski style* (using nonmonotonic consequence closure $\mathcal{C}$).

In fact, they are exactly expressible each other by
$\mathcal{C}(\Gamma) = \{A : \Gamma \mid\sim A\}$.
In this paper, we will investigate three possible ways to construct a natural deduction system for nonmonotonic reasoning.

1. A system with pure nonmonotonic inference rules;

2. A system with nonmonotonic inference rules plus a non monotonic tautology base; and

3. A Conditional Knowledge Base which is a system with nonmonotonic inference rules plus a nonmonotonic conditional assertion base.

We show that inferences in each system can be reduced to the ones in classical logic. We will see that this result provide us more methodological perspective than its technical implication.

This paper is organized as follows, in the next section we will provide some basic concepts and some inference rules for non monotonic reasoning put forth by the literature. In section 3 will discuss the criteria for natural deduction system for nonmonotonic reasoning. In section 4 we discuss **RN** frame work which form the basis for our discussion, In section 5 we will discuss the three possible solutions for the question posed. In section 6 we will conclude this paper with the results.

2 Definitions and concepts

We will restrict the language of the indented system within a propositional language $\mathcal{L}$. Elements of $\mathcal{L}$ are called formulas which are denoted by A,B,C. Sets of formulas are denoted by Γ, Δ,etc. There are two inference relations between premises (on the left) and conclusions(on the right):
$\vdash$, denoting the classical propositional derivability, and
$\mid\sim$, used for nonmonotonic inference relation. The Tarski-style's consequence operator for classical derivability is denoted by Cn.

A set Γ of formulas is said to be closed if $\Gamma = Cn(\Gamma)$.
We write $\Gamma \mid\sim (\vdash) \Delta$ as the abbreviation of $\Gamma \mid\sim (\vdash) A$ for all $A \in \Delta$ (Δ may be empty).
$\Gamma \not\mid\sim A$ indicates that $\Gamma \mid\sim A$ does not hold. If $\emptyset \mid\sim A$ is derived from a deductive system, where $\emptyset$ denotes the empty set, then we call A a *non monotonic tautology* in the system.

2.1 Nonmonotonic inference rules

We list some nonmonotonic inference rules which are put forward in the literature. We will present the infinite version of them and most of them are written in Gentzen style. This list starts with the three Gabbay's rules and is followed :

1 Reflexivity:
If $A \in \Gamma, \Gamma \mid\sim A$.

2 Cut:
If $\Gamma \mid\sim A$ *and* $\Gamma, A \mid\sim B$, *then* $\Gamma \mid\sim B$.

3 Restricted Monotonicity:
If $\Gamma \mid\sim A$ *and* $\mathcal{T} \mid\sim B$, *then* $\Gamma, A \mid\sim B$.

4 Supraclassicality:
If $\Gamma \vdash A$, *then* $\Gamma \mid\sim A$.

5 Consistency Preservation:
If $\Gamma \mid\sim A$, *then* $\Delta \vdash A$.

6 Weak Transitivity:
If $\Gamma \mid\sim \Delta \vdash A$, *then* $\Gamma \mid\sim A$.

7 Cumulative Transitivity:
If $\Gamma \mid\sim \Delta$ *and* $\Gamma \cup \Delta \mid\sim A$, *then* $\mathcal{T} \mid\sim A$.

8 Cautious Monotony:
If $\Gamma \mid\sim \Delta$ *and* $\mathcal{T} \mid\sim A$, *then* $\Gamma \cup \mathcal{P} \mid\sim A$.

9 Reductio ad Absurdum:
If $\Gamma, A \mid\sim B, \neg B$, *then* $\Gamma \mid\sim \neg A$.

10 Deduction Theorem:
If $\Gamma, A \mid\sim B$, *then* $\mathcal{T} \mid\sim A \rightarrow B$.

11 Modus Ponens:
If $\Gamma \mid\sim A \rightarrow B$ *and* $\Gamma \mid\sim A$, *then* $\Gamma \mid\sim B$.

12 Reciprocity:
If $\Gamma \mid\sim \Delta$ *and* $\mathcal{P} \mid\sim \Gamma$, *then* $\Gamma \mid\sim A$ *if and only if* $\Delta \mid\sim A$.

13 Left Logical Equivalence:
If $\Gamma \dashv\vdash \Delta$, *then* $\Gamma \mid\sim A$ *if and only if* $\mathcal{P} \mid\sim A$.

14 Right Logical Equivalence:
If $A \dashv\vdash B$, *then* $\Gamma \mid\sim A$ *if and only if* $\Gamma \mid\sim B$.

15 Distribution:
If $\Gamma \cup \Delta_1 \mid\!\sim A$, $\mathcal{T} \cup \Delta_2 \mid\!\sim A$, *then* $\Gamma \cup (\Delta_1 \bigvee \Delta_2) \mid\!\sim A$, *where* $\Delta_1 \bigvee \Delta_2 = \{A \vee B : A \in \Delta_1$ *and* $\Delta_2\}$.

16 (Loop):
If $\Gamma_0 \mid\!\sim \Gamma_1 \mid\!\sim \cdots \mid\!\sim \mathcal{T}_n \mid\!\sim \Gamma_0$ *(*$n > 0$*), then for any* $i, j \in \{0, 1, \cdots, n\}$, $\Gamma_i \mid\!\sim A$ *iff* $\Gamma_j \mid\!\sim A$.

Although Gentzen style is intuitive and readable, some rules are easily appreciated in Tarski style such as:

17 Absorption:
$Cn\mathcal{C} = \mathcal{C} = \mathcal{C}Cn$.

18 Infinite Conditionalization:
$\mathcal{C}(\Gamma \cup \Delta) \subseteq Cn(\mathcal{T} \cup \mathcal{C}(\Delta))$.

19 Rational Monotony:
IF $\Delta \cup \mathcal{C}(\Gamma) \neq \mathcal{L}$, *then* $\mathcal{C}(\Gamma) \subseteq \mathcal{C}(\Gamma \cup \Delta)$.

The following two rules are the nonmonotonic version of compactness:

20 Supracompactness:
$\Gamma \mid\!\sim A$ *iff there exists a finite subset* Γ_0 *of* Γ *such that for any set of formulas* Δ, $\Gamma \mid\!\sim A$ *implies* $\mathcal{T}_0 \cup \Delta \mid\!\sim A$.

21 Finite Supracompactness:
$\Gamma \mid\!\sim A$ *iff there exists a finite subset* Γ_0 *of* Γ *such that* $\Gamma_0 \cup \mathcal{T}' \mid\!\sim A$ *for every finite subset* Γ' *of* $Cn(\Gamma)$.

Obviously this is not a complete list of nonmonotonic inference rules. However, most of them which were suggested as a rule for general nonmonotonic reasoning in the literature are in the list except those which

- obviously can be derived from the list, such as *Proof by Cases*, *Negation Rationality* and *Weak Contraposition* (see [17][4]);

- if added, will cause the inference relation collapsing to monotony, such as *Monotonicity*, *Transitivity* and *Contraposition* (see [14][4]).

- were proposed for special interest of nonmonotonic inference, such as *Determinacy Preservation*, *Antitonicity*, *Rational Contraposition* and *Weak Determinacy*(see [17][11][2]).

Note that most of the rules are the counterparts or some sort of weakening of classical rules. However, *Supraclassicality* seems to be the only one which shows that nonmonotonic derivability is stronger than the classical one. This rule reflects that nonmonotonic reasoning has the extra capability to "jump to conclusions".

We remark that these inference rules are by no means independent. We have

Proposition 1 *If a nonmonotonic inference relation* $\mid\!\sim$ *satisfies supraclassicality, consistency preservation, weak transitivity, infinite conditionalization, rational monotonicity , and finite supracompactness, then all the others in the list can be derived.*

Proof: According to Lemma 2.2, 2.3, 2.4 and Theorem 2.5 in [22], all the rules except (16) and (17) are derivable from this set of rules. According to Observation 2.21 and 2.28 in [17], (16) is implied by *Reflexivity*(1) , *Supraclassicality*(4) , *Cumulative Transitivity*(7) , *Cautious Monotony*(8) and *Distribution*(15). (16) *follows from Reflexivity*(1), *Supraclassicality*(4), *Weak Transitivity*(6), *Cumulative Transitivity*(7)*and Cautious Monotony*(8)□

3 Criteria for nonmonotonic Natural Deductive System

Since the list in section 2 embraces the most representative non monotonic inference rules, one idea is to select some inference rules from the list to constitute a natural deduction system of nonmonotonic reasoning. Then what is the criteria for such a selection. In other words, what is the baseline for a natural deduction system of nonmonotonic inference?

1 A natural deduction system should only consist of inference rules.

2 it should be nonmonotonic, that is, the rule:
If $\Gamma \mid\!\sim A$, *then* $\mathcal{T} \cup \Delta \mid\!\sim A$ *(Monotonicity)* should not be includd in the deductive system.

3 It should be consistent.

4 It should be able to *jump to conclusions*.

Here a set of nonmonotonic inference rules is *consistent*, if for any classically consistent set of formulas Γ, there is no formula $A \in \mathcal{L}$ such that both $\Gamma \mid\!\sim A$ and $\Gamma \mid\!\sim \neg A$ can be entailed in the system. This requires that the non monotonic inference does not introduce new contradictions.

In other words, the nonmonotonic inference should entail no less consequences than classical one from same premises and when it is equipped with an extra commonsense knowledge base it can entail more. This qualification is apt to be confused with the monotonicity. For instance, it seems to require that if $penguin$ entails can_fly classically, then $penguin$ should entail can_fly nonmonotonically as well. This is a misunderstanding. nonmonotonic reasoning distinguishes observed facts(hard facts) and commonsense knowledge (or background knowledge).

We say $\Gamma \vdash A$ to mean that A is the classical consequence of the set Γ of hard facts (disregarding of commonsense knowledge), whereas $\Gamma \mid\!\sim A$ means A can be derived from Γ under a support of some commonsense knowledge base. Therefore it is reasonable to require that the nonmonotonic inference system can entail more consequences from the same promises than the classical logic because the former is generally equipped with an extra commonsense knowledge base.

4 Rational nonmonotonic Frame

Now we consider the possible constitutions of nonmonotonic natural deduction systems. In general, the more rules we have, the more consequences we can derive[1]. Thus our first choice is to select a set of rules so that it can derive as many nonmonotonic inference rules as possible. By proposition 1, if we choose the six rules listed in that proposition, we will be able to derive all the non monotonic inference rules we listed in section 2.

As defined in [22], a structure $(\mathcal{L}, \mid\!\sim)$ was called a *rational nonmonotonic frame*, where $\mathcal{L}$ is a propositional language and $\mid\!\sim$ is an inference relation on $2^{\mathcal{L}} \times \mathcal{L}$ which satisfies *Supraclassicality, Consistency Preservation, Weak Transitivity, Infinite Conditionalization* and *Rational Monotonicity*. It is *finite supracompact* if the nonmonotonic inference relation satisfies further *Finite Supracompactness.*

Such a system is called a $\mathbf{RN}$ *frame* and the associated inference rules are referred to the *basic* $\mathbf{RN}$ *rules.* It is not hard to see that an inference relation in a $\mathbf{RN}$ frame corresponds to the rational (nonmonotonic) inference operation $\mathcal{C}$ in terms of [Freund and Lehmann 94] except that the later does not require $\mathcal{C}$ to satisfy *Consistency Preservation* and *Finite Supracompactness.*

For any $\mathbf{RN}$ frame $(\mathcal{L}, \mid\!\sim)$, if $\phi \mid\!\sim A$ can be derived from the frame, where ϕ *denotes* the empty set, then we call A a *nonmonotonic tautology* in the frame.

In order to model a $\mathbf{RN}$ frame, [22] introduced a relationship between the $\mathbf{RN}$ derivability and the general (multiple) belief revision ([21] [23]). The idea is to make the following correspondence between nonmonotonic inference and belief revision:

$$\Gamma \mid\!\sim A \text{ iff } A \in K \otimes \mathcal{T}$$

[1]*Rational Monotony* is an exception.

where K is the set of nonmonotonic tautologies, that is, $K = \{B : \phi \mid\!\sim B\}$. It is easy to see that this is a generalization of the relationship between nonmonotonic reasoning and (single) belief revision given in [18] [9], where K was interpreted as the background knowledge of nonmonotonic reasoning. More precisely, from [22] we have

Theorem 1 *Let K be a consistent closed set in language $\mathcal{L}$ and $\otimes$ a general revision function over K. Let $\mid\!\sim \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ be a relation which satisfies $\Gamma \mid\!\sim A$ iff $A \in K \otimes \Gamma$. for any $\Gamma \subseteq \mathcal{L}$ and $A \in \mathcal{L}$. Then $(\mathcal{L}, \mid\!\sim)$ is a $\mathbf{RN}$ frame. Conversely, if $(\mathcal{L}, \mid\!\sim)$ is a $\mathbf{RN}$ frame, then there exists a general belief revision function $\otimes : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ such that for any $\mathcal{T} \subseteq \mathcal{L}$, $K \otimes \Gamma = \{A \in \mathcal{L} : \mathcal{T} \mid\!\sim A\}$, where $K = \{B \in \mathcal{L} : \phi \mid\!\sim B\}$*□

This result gives us a way to think about nonmonotonic reasoning by belief revision. We will find that reasoning in belief revision is sometimes much easier than reasoning in non monotonic inference rules.

We remark that a $\mathbf{RN}$ frame must be "very strong" because all the rules in our list are derivable in any $\mathbf{RN}$ frame**.** In fact, the logical system **P,** which is the strongest one other than monotonic logic among the systems introduced in [14], Poole's *theorist* without constraints ([20] [5]), Nebel's *ranked default theory* ([19]) and its generalization *syntax-independent default-theory*([24]) are subsystems of some $\mathbf{RN}$ frames[2].

5 nonmonotonic Natural Deduction

In this section, we will consider several possible constitutions of natural deduction systems of nonmonotonic reasoning. The first choice is using only the basic $\mathbf{RN}$ rules to construct such a system.

5.1 Pure nonmonotonic inference rules

Let Ω denote the deductive system which is composed of all the basic $\mathbf{RN}$ rules. Now we examine whether Ω satisfies all the criteria given in section 3. No problem for the first one. For the second one, we need to show that the *monotonicity* is independent with all the rules of Ω, that is

Proposition 2 *The Monotonicity is not derivable in Ω.*

[2]Here a non monotonic system $(\mathcal{L}, |\!\sim)$ being a *subsystem* of a $\mathbf{RN}$ frame $(\mathcal{L}, |\!\sim_{RN})$ means that for any Γ and A, if $\Gamma |\!\sim A$, then $\Gamma |\!\sim_{RN} A$.

Proof: According to Theorem 1, it is equivalent to prove that the following postulate is independent with the nine postulates for the general belief revision (see [21][23]):

$(\otimes M)$ $\quad K \otimes F_1 \subseteq K \otimes (F_1 \cup F_2)$

Suppose that the nine postulates for the general belief revision implies $(\otimes M)$. Consider a propositional language $\mathcal{L}$ with at least two propositional letters, say p and q. Let $K = Cn(\{p,q\})$ and $\Sigma = (K, \mathcal{P}, <)$ be a nice-ordered partition of K, where $\mathcal{P} = \{Cn(\{q\}), Cn(\{p,q\}) \backslash Cn(\{q\})\}$. According to Definition 4.5 in [21], we can have a general contraction function $\ominus$ over K such that

$K \ominus \{\neg p\} = Cn(\{q\})$

$K \ominus \{\neg p, \neg q\} = Cn(\{p \vee \neg q, \neg p \vee q\})$.

Therefore, by *Levi identity*, we have

$K \otimes \{\neg p\} = Cn(\{\neg p, q\})$

$K \otimes \{\neg p, \neg q\} = Cn(\{\neg p, \neg q\})$

Therefore $K \otimes \{\neg p\} \not\subseteq K \otimes \{\neg p, \neg q\}$ even though $\{\neg p\} \subseteq \{\neg p, \neg q\}$. This contradicts $(\otimes M)\Box$

The consistency of Ω is implied by Theorem 1. So the remainder is to check if Ω can help us to jump to conclusions.

Observation 1 *For any* $\Gamma \subseteq \mathcal{L}$ *and* $A \in \mathcal{L}$, $\Gamma {\mid\!\sim}_\Omega A$ *iff* $\Gamma \vdash A$.

Proof: "$\Leftarrow$" is implied by the *Supraclassicality*.
"$\Rightarrow$" Let $K = Cn(\phi)$ and $\otimes$ be any general revision function over K. Then according to Theorem 1, there exists a **RN** frame $(\mathcal{L}, {\mid\!\sim})$ such that

$$A \in K \otimes \Gamma \text{ iff } \Gamma {\mid\!\sim} A$$

Since $K \otimes \Gamma = Cn(\Gamma)$, $A \in K \otimes \Gamma$ implies $\Gamma \vdash A$. So $\Gamma {\mid\!\sim} A$ implies $\Gamma \vdash A$. Suppose that $\Gamma {\mid\!\sim}_\Omega A$ can be derived in Ω, then it can also be derived in $(\mathcal{L}, {\mid\!\sim})$, i.e. $\Gamma {\mid\!\sim} A$. Therefore $\Gamma \vdash A$. $\Box$

This observation says that all the inference relation which are entailed by Ω are exactly the same as the ones which are derivable in the classical logic[3]. Therefore Ω does not qualify as a natural deduction system for nonmonotonic reasoning because it can not help us jump to conclusions, let along any subsystem of Ω.

Since Ω has included almost all inference rules we have had, this result showed us that we might not have a natural deduction system with the classical style (consisting of pure inference rules) for nonmonotonic reasoning. This does not surprise us at all because all the nonmonotonic inference rules except *Supraclassicality* are some sort of weakening of the classical logic rules. In fact, we should not expect Ω to jump to conclusions in absence of commonsense knowledge. However, this result tells us that we may have not a "classical style" natural deduction system for nonmonotonic reasoning which involves only "general" inference rules.

[3] Note that this does not mean that ${\mid\!\sim}$ in Ω is equivalent to the classical derivability $\vdash$ (so does not contradicts Proposition 2).

5.2 Nonmonotonic tautologies

As we remarked before, pure nonmonotonic inference rules are too weak to compose a deduction system for jumping to conclusions. We have to explore some *non-classical style* of deduction systems for nonmonotonic reasoning. Our first try to this end is to upgrade Ω by equipping it with a commonsense knowledge base or background knowledge base.

Normally a knowledge base $\mathcal{B}$ is expressed by a set of sentences. However, a natural deduction system can only be composed of inference rules. Thus we need to convert each sentence into an inference rules. A natural idea to do this is to convert each sentence $A \in \mathcal{B}$ into a nonmonotonic tautology $\phi {\mid\!\sim} A$.

More precisely, let $\Omega_{\mathcal{TB}}$ be a natural deduction system which consists of all the inference rules of Ω and the nonmonotonic tautologies base $\mathcal{TB} = \{\phi {\mid\!\sim} A : A \in \mathcal{B}\}$.

Now we check if $\Omega_{\mathcal{TB}}$ can be a natural deduction system for nonmonotonic reasoning with respect to the background knowledge base $\mathcal{B}$ (Notice the difference between $\mathcal{B}$ and $\mathcal{TB}$).

The independency of monotonicity is similar to the theorem 2 (The only difference is when we construct the belief set K in the proof of the proposition 2, we need to let $\mathcal{B} \subseteq K$). According to Theorem 1, it is not very hard to prove that $\Omega_{\mathcal{TB}}$ is consistent if and only if $\mathcal{B}$ is consistent. Finally, we check if the extra rules can help us jumping to conclusions. To this end, let's begin with the old Tweety example.

Example 1 *Suppose that we have a background knowledge base* $\mathcal{B} = \{tweety \to bird, tweety \to \neg fly, bird \to fly\}$. *We express it into the following nonmonotonic tautology base* $\mathcal{TB}$*:*

(1) ${\mid\!\sim} tweety \to bird$

(2) ${\mid\!\sim} tweety \to \neg fly$

(3) ${\mid\!\sim} bird \to fly$

Consider the following three inference relations:

I*:* $fly {\mid\!\sim} \neg tweety$

II*:* $tweety {\mid\!\sim} \neg fly$

III: $tweety \wedge bird \mid\!\sim \neg fly$
Are they derivable in $\Omega_{\mathcal{TB}}$?

Although all of them are intuitively true, only the first one is derivable. Even though, to verify or refute them is not an easy work, especially, to refute something. The following observation is generally quite helpful.

Observation 2 *For any given consistent set $\mathcal{B}$ of sentences, let $\Omega_{\mathcal{TB}}$ be the associated nonmonotonic inference system. Then for any $\Gamma \subseteq \mathcal{L}$ and $A \in \mathcal{L}$,*
i) *if $\mathcal{B} \cup \Gamma$ is inconsistent, then $\Gamma \mid\!\sim_{\Omega_{\mathcal{TB}}} A$ iff $\Gamma \vdash A$.*
ii) *if $\mathcal{B} \cup \Gamma$ is consistent, then $\Gamma \mid\!\sim_{\Omega_{\mathcal{TB}}} A$ iff $\mathcal{B} \cup \Gamma \vdash A$.*

Proof: First, we prove the easy direction (from the right hand side to the left hand side) for each case.
For the case i),if $\Gamma \vdash A$, then, by *Supraclassicality*, $\Gamma \mid\!\sim_{\Omega_{\mathcal{TB}}} A$.
For the case ii), suppose that $\mathcal{B} \cup \Gamma$ is consistent and $\mathcal{B} \cup \Gamma \vdash A$.
On one hand, $\mathcal{B} \cup \Gamma$ is consistent implies that for any $A_1, \cdots, A_n \in \Gamma$, $\phi \mid\!\not\sim_{\Omega_{\mathcal{TB}}} \neg(A_1 \wedge \cdots \wedge A_n)$ (otherwise, contradicting the consistency of $\Omega_{\mathcal{TB}}$). On the other hand, $\mathcal{B} \cup \Gamma \vdash A$ implies there exist $B_1, \cdots, B_n \in \Gamma$ such that $\mathcal{B} \vdash (B_1 \wedge \cdots \wedge B_n) \rightarrow A$. According to *Weak Transitivity*, $\phi \mid\!\sim_{\Omega_{\mathcal{TB}}} (B_1 \wedge \cdots \wedge B_n) \rightarrow A$. Then by *Rational Monotony*, $\Gamma \mid\!\sim_{\Omega_{\mathcal{TB}}} (B_1 \wedge \cdots \wedge B_n) \rightarrow A$. Since $\Gamma \vdash B_1 \wedge \cdots \wedge B_n$, we obtain by *Supraclassicality* and *S* that $\Gamma \mid\!\sim_{\Omega_{\mathcal{TB}}} A$.

Now we go for the other direction of each case. Let $K = Cn(\mathcal{B})$ and $\ominus$ be a full meet contraction function(see [23]), that is, for any set $\mathcal{T}$ of sentences $K \ominus \Gamma = \bigcap (K \| \Gamma)$[4].

If $K \cup \Gamma$ is consistent, $K \ominus \Gamma = K$; otherwise, we prove that $K \ominus \Gamma = Cn(\Gamma) \cap \Gamma$.

In fact, assume that $A \in \bigcap (K \| \Gamma)$. If $\Gamma \not\vdash A$, then $\Gamma \cup \{\neg A\}$ is consistent. So for any $B \in K$, $\{\neg A \vee B\} \cup \Gamma$ is consistent and $\neg A \vee B \in K$. Then there exists $K' \in K \| \Gamma$ such that $\neg A \vee B \in K'$. Since $A \in \bigcap (K \| \Gamma)$, $A \in K'$. Then $B \in K'$, this means $K' = K$, which contradicts the fact that $K \cup F$ is inconsistent. To show the converse, assume that $A \in K$ and $\Gamma \vdash A$. If $A \notin \bigcap (K \| \Gamma)$, then there exists $K' \in K \| \Gamma$ such that $A \notin K'$. Therefore $K' \cup \Gamma \cup \{A\}$ is inconsistent, which contradicts the fact that $K' \cup \Gamma$ is consistent.

Now let $\otimes$ be the corresponding revision function of $\ominus$, i.e., $K \otimes \Gamma = K \ominus \Gamma + \Gamma$. Thus we have that:
i) $K \otimes \Gamma = K + \Gamma$, if $K \cup \Gamma$ is consistent;
ii) $K \otimes \Gamma = Cn(\Gamma)$, otherwise.

It is not hard to prove that a revision function defined from a full meet contraction satisfies all the nine postulates for the general belief revision. Thus, according to Theorem 1, there exists a $\mathbf{RN}$ frame $(\mathcal{L}, \mid\!\sim_{RN})$ such that $\Gamma \mid\!\sim_{RN} A$ iff $A \in K \otimes \Gamma$ as well as $K = \{A : \phi \mid\!\sim_{RN} A\}$.

If $\mathcal{B} \cup \Gamma$ is consistent, then $K \cup \Gamma$ is consistent. So $K \otimes \Gamma = K + \Gamma$. It follows that

$$\Gamma \mid\!\sim_{RN} A \text{ iff } A \in K \otimes \Gamma \text{ iff } A \in K + \Gamma \text{ iff } K \cup \Gamma \vdash A \text{ iff } \mathcal{B} \cup \Gamma \vdash A$$

If $\mathcal{B} \cup \Gamma$ is inconsistent, then $K \otimes \Gamma = Cn(\Gamma)$. So $\Gamma \mid\!\sim_{RN} A$ iff $A \in Cn(\Gamma)$ iff $\Gamma \vdash A$.

Because $(\mathcal{L}, \mid\!\sim_{RN})$ possesses all the rules of $\Omega_{\mathcal{TB}}$, $\Gamma \mid\!\sim_{\Omega_{\mathcal{TB}}} A$ implies $\Gamma \mid\!\sim_{RN} A$. So $\mathcal{B} \cup \Gamma \vdash A$ when $\mathcal{B} \cup \Gamma$ is consistent, otherwise, $\Gamma \vdash A$ □

This observation says that only if the premises of an inference is consistent with the background knowledge base, which is generally considered to be the trivial case of nonmonotonic reasoning, the nonmonotonic inference could entail more than classical logic. When the promises of an inference are inconsistent with the background knowledge base (the non-trivial case), the nonmonotonic inference relation is equivalent to the classical derivability. Moreover, no matter whether promises of an inference are consistent with the background knowledge base, nonmonotonic derivation can always be reduced to the one of classical logic.

Let's go back to the inference relations (I), (II) and (III) we considered in Example 1. Since $\mathcal{B} \cup \{fly\}$ is consistent and $\mathcal{B} \cup \{fly\} \vdash \neg tweety$, (I) is derivable. For (II), since $\mathcal{B} \cup \{tweety\}$ is inconsistent and $tweety \not\vdash \neg fly$, (II) is not derivable. Similar to (III). It is obvious that the "nonmonotonic inference by the classical logic" is much easier than inference by directly using nonmonotonic inference rules.

inference and a reduced to the classical one. than a $\mathbf{RN}$ frame,

5.3 Conditional Knowledge Base

We have investigated the inference power of pure nonmonotonic inference rules and these rules plus a set of nonmonotonic tautologies. In both cases, the nonmonotonic inference can easily be reduced to the classical logic. In this section, we consider a more general case that the commonsense knowledge is expressed by a set of conditional assertions.

[4] $K' \in K \| \Gamma$ if and only if
1. $K' \subseteq K$;
2. $\Gamma \cup K'$ is consistent, and
3. $\forall K'' \subseteq K (K' \subset K'' \rightarrow K' \cup F$ is inconsistent). (see [23] for details.)

In [15], a nonmonotonic inference relation in the form $A|\!\sim B$ is called a *conditional assertion*. A set of conditional assertions is called a *conditional knowledge base*.

Let's see if there is any difference between a nonmonotonic tautology and a conditional assertion.

Example 2 *Suppose that the background knowledge base in example 1 is written in the following conditional knowledge base $\mathcal{CB}$,*
(1) $tweety|\!\sim bird$
(2) $tweety|\!\sim \neg fly$
(3) $bird|\!\sim fly.$

Then all the inference relations (I), (II) and (III) considered in Example 1 become derivable.

Let $\mathcal{B}$ be a set of sentences and $\mathcal{CB}$ a conditional knowledge base. If

$$\mathcal{B} = \{A \to B : A|\!\sim B \in \mathcal{CB}\} \quad (1)$$

we call $\mathcal{CB}$ a conditional knowledge base *based on* $\mathcal{B}$

For any given conditional knowledge base $\mathcal{CB}$, let $\Omega_{\mathcal{CB}}$ denote the deductive system which consists of all the inference rules of Ω and the rules in $\mathcal{CB}$. We now investigate the inference power of $\Omega_{\mathcal{CB}}$.

First, it is easy to see that $\Omega_{\mathcal{CB}}$ is no weaker than $\Omega_{\mathcal{TB}}$, where $\mathcal{TB} = \{\phi|\!\sim A \to B : A|\!\sim B \in \mathcal{CB}\}$. Secondly, we know that any finite inference relation[5] can be expressed by a conditional assertion. According to *Finite Supracompactness*, an infinite inference relation can be reduced to the finite ones, and therefore, any nonmonotonic inference relation is expressible by conditional assertions. In this sense, the inference in $\Omega_{\mathcal{CB}}$ can be transferred to the inference about conditional assertions, which has been investigated by Lehmann and Magidor [15].
So it seems now that we are able to answer the question raised in the introduction of the paper: *a natural deduction system for nonmonotonic reasoning is the* **RN** *rules plus a particular conditional knowledge base.* Given a conditional knowledge base $\mathcal{CB}$, according to Lehmann and Magidor's argument, all the conditional assertions which can be derived from $\mathcal{CB}$ should be in the so-called *rational closure* $\overline{\mathcal{CB}}$ of $\mathcal{CB}$ (see [15]). If we take Lehmann and Magidor's argument as an answer to our question, we would conclude that $A|\!\sim B$ is derivable in $\Omega_{\mathcal{CB}}$ iff $A|\!\sim B \in \overline{\mathcal{CB}}$. The general case $\Gamma |\!\sim A$ can be dealt with by *Finite Supracompactness* . However, suppose that we are given a commonsense knowledge base which is a set of conditional expressions of the form: *normally, if A then B*. Should we express it as a nonmonotonic tautology "$\emptyset|\!\sim A \to B$" or as a conditional assertion "$A|\!\sim B$"? Example 1 and 2 have shown the big difference between them and we know that generally the later is stronger than the former. A question is *what is the extra information attached by a conditional assertion?*

Before we answer this question, let's show a lemma.

Lemma 1 *Let $K = \{A : \phi|\!\sim_{\Omega_{\mathcal{CB}}} A\}$. Then $K = Cn(\mathcal{B})$.*

proof: Since $\mathcal{CB} \subseteq \overline{\mathcal{CB}}$, by *Deduction Theorem*([10]) we have $Cn(\mathcal{B}) \subseteq K$(see Equation (1)). For the other direction, assume $\phi|\!\sim A \in \overline{\mathcal{CB}}$. According to lemma 5.15 in [Lehmann and Magidor 92], $\phi|\!\sim A$ can be preferentially entailed by $\mathcal{CB}$. By lemma 5.21 in [Lehmann and Magidor 92], we have $A \in Cn(\mathcal{B})$. $\Box$

The following observation shows that the extra information attached by a conditional assertion is an ordering on background knowledge.

Observation 3 *For any conditional knowledge base $\mathcal{CB}$ based on $\mathcal{B}$, if its rational closure exists and satisfies Consistency Preservation, then $\mathcal{CB}$ uniquely determines an ordering $<$ on $Cn(\mathcal{B})$ such that for any finite set Γ and a sentence A, $\Gamma|\!\sim_{\Omega_{\mathcal{CB}}} A$ iff $\mathcal{B} \cup \Gamma \vdash A$ and when $\mathcal{B} \cup \Gamma$ is inconsistent, then either $\Gamma \vdash A$ or $\neg(\wedge\Gamma) < (\wedge\Gamma) \to A$.*

Proof: In the case that $\mathcal{B} \cup \Gamma$ is inconsistent, since $\Gamma|\!\sim_{\Omega_{\mathcal{CB}}} A$ implies $\Gamma|\!\sim_{\Omega_{\mathcal{TB}}} A$, by observation 2 we know that $\Gamma|\!\sim_{\Omega_{\mathcal{CB}}} A$ iff $\mathcal{B} \cup \Gamma \vdash A$. Now we assume that $\mathcal{B} \cup \Gamma$ is inconsistent.

Let $\overline{\mathcal{CB}}$ be the rational closure of $\mathcal{CB}$. According to the definition of rational closure, $\overline{\mathcal{CB}}$ is a rational consequence relation. In addition, $\overline{\mathcal{CB}}$ satisfies *Consistency Preservation*. Thus by *finite supracompactness* $\overline{\mathcal{CB}}$ uniquely determines a **RN** frame $(\mathcal{L}, |\!\sim)$. Then according to Theorem 1, it again uniquely determines a general revision function, specially, also an AGM revision function '$*$' over the belief set $K = \{A : \phi|\!\sim A\}$ such that

$$\Gamma|\!\sim A \text{ iff } A \in K * (\wedge\Gamma) \quad (2)$$

Suppose that '$-$' be the associated contraction function of '$*$'. According to [7], '$-$' uniquely determines an epistemic entrenchment ordering $<$ such that

$$B \in K - A \text{ iff } B \in K \text{ and either } \vdash A \vee B \text{ or } A < A \vee B$$

Thus $A \in K * (\wedge\Gamma)$ iff $(\wedge\Gamma) \to A \in K$ and either $\vdash (\wedge\Gamma) \to A$ or $\neg(\wedge\Gamma) < (\wedge\Gamma) \to A$.

By Lemma 1, we have that $(\wedge\Gamma) \to A \in K$ iff $(\wedge\Gamma) \to A \in Cn(\mathcal{B})$ iff $\mathcal{B} \cup \Gamma \vdash A$. Then by (2) we obtain $\Gamma|\!\sim_{\Omega_{\mathcal{CB}}} A$

[5] A finite inference relation means an inference relation with only finite premises.

iff $\Gamma \mid\!\sim B \in \overline{\mathcal{CB}}$ iff $\Gamma \mid\!\sim A$ iff $A \in K * (\wedge\Gamma)$ iff $\mathcal{B} \cup \Gamma \vdash A$ and either $\vdash (\wedge\Gamma) \to A$ or $\neg(\wedge\Gamma) < (\wedge\Gamma) \to A$ □

According to the proof of the observation, the ordering corresponding to the conditional knowledge base can be anyone like epistemic entrenchment [7], nice-ordered partition [23] and adventurousness [15]. This observation shows that the inference in $\Omega_{\mathcal{CB}}$ can be reduced to the classical logic and comparison of ordering (infinite inference can be transferred into the finite case by finite supracompactness). In other words, there are two ways of performing nonmonotonic inference: by nonmonotonic rules or by classical logic plus ordering comparing. Although it is hard to say that which way does human being is using, the following example shows that the second way may be relatively easier.

Example 3 Consider Example 1 again. If Tweety is a penguin, we may give more preference (more entrenchment) to $tweety \to bird$ and $tweety \to \neg fly$ than to $bird \to fly$. By comparing Example 1 and Example 2, we may think that the conditional knowledge base in Example 2 seems better to reflect our preference than the nonmonotonic tautology base. However, if we are told that Tweety is a bad painted cartoon character, we might give less preference to $tweety \to bird$ than $tweety \to \neg fly$ and $bird \to fly$. The problem is which conditional knowledge base would reflect such a preference?

We may find that it is generally not very easy to express a preference on our commonsense knowledge by a conditional knowledge base. We seem to have more intuition on the ordering of knowledge base rather than a conditional knowledge base.

6 Conclusion and Discussion

We have considered three possible constructions of a natural deduction system for nonmonotonic reasoning:

1) pure nonmonotonic inference rules.

2) nonmonotonic inference rules plus a set of nonmonotonic tautologies.

3) nonmonotonic inference rules plus a set of conditional assertions.

The first one has been proved to be too weak “to jump to conclusions”. The second one could entail more only when the promises of an inference relation is consistent with the commonsense knowledge base, which is generally viewed as the trivial case in nonmonotonic reasoning. The last one seems to be the only interesting one. However, we have proved that each of these approaches has its counterpart in classical logic. We can always transfer a nonmonotonic inference into an inference in classical logic. Then comes an interesting question: which way for nonmonotonic deduction is more preferable: *nonmonotonic inference rules plus a conditional knowledge base* or *classical inference rules plus an ordering on the knowledge base*?

There seems to be no simple answer to it. Reasoning in the latter way is relatively simple and practically feasible, but not as mathematically elegant as in the former way. Obviously, a natural deduction system of nonmonotonic reasoning has its theoretical interest. First, it is a purely logical approach. The deduction in this approach is of the classical style of proof theory. Secondly, we can choose different combination of inference rules or invent new inference rules to endow the nonmonotonic inference relation $\mid\!\sim$ with different meaning. Such flexibility gives more logical meaning than the flexibility of an ordering on knowledge base. However, this work has revealed some of its disadvantages:

- The nonmonotonic natural deduction is not as easy to use as its classical counterpart, especially the use of the rules like *Rational Monotony*. It is very hard to image a tableau-like theorem prover of nonmonotonic reasoning which is based on the natural deduction rather than its classical counterpart.

- The natural deduction of nonmonotonic reasoning is no more necessarily confluent, i.e., using rules in different order would lead to different results.

- It is generally not very easy to distinguish the connotation between a conditional assertion $A \mid\!\sim B$ and nonmonotonic tautology $\emptyset \mid\!\sim A \to B$. However, such distinction is crucial in nonmonotonic reasoning.

- It is also not very clear how to use conditional assertions to express a preference of commonsense knowledge.

We by no means follow up all possible constructions of natural deduction systems for nonmonotonic reasoning. For instance, we have not consider the case that the conditional knowledge base includes negative conditional assertions ($A \mid\!\not\sim B$, see [1]). Additionally, all results we obtained in the paper are based on our criteria for nonmonotonic natural deduction and the nonmonotonic rules we listed. We could release these restrictions or invent new inference rules for nonmonotonic reasoning (or taking account of those which are not included in our list). However, we believe that our approach and the negative factors above are applicable in more general cases.

Nonmonotonic reasoning has been investigated by AI researchers for over twenty years. Hundreds of logical systems have been proposed with varying degree of success to capture the characteristics of human's nonmonotonic reasoning. Most of them are kinds of dissimilation or extension of the classical logic. Clearly, we can not or need not develop a nonmonotonic counterpart for each approach of classical logic. This paper might cast a shadow on the attempt of nonmonotonic natural deduction systems. Natural deduction would be theoretically attractive but might not be practically tractable for nonmonotonic reasoning.

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