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How to efficiently allocate houses under price controls?[☆]

Tommy Andersson[a,*], Zaifu Yang[b], Dongmo Zhang[c]

[a] Department of Economics, Lund University, Sweden
[b] Department of Economics and Related Studies, University of York, York, UK
[c] Intelligent Systems Laboratory, University of Western Sydney, Australia

HIGHLIGHTS

- We examine a practical housing market in the presence of price controls.
- We demonstrate how houses can be efficiently allocated through a price system.
- We show that the auction by Talman and Yang (2008) converges to a core outcome.

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ABSTRACT

We examine a housing market with price controls and show how the allocation problem can be solved through a price system. We demonstrate that the auction of Talman and Yang (2008) always generates a core allocation, thus resulting in a Pareto efficient and stable outcome.

1. Introduction

We study a practical housing market where houses are rented to tenants.[1] Each tenant has valuations on houses and is interested in renting at most one house. The landlord has a reservation rent for her houses and is not willing to let the house to any tenant if the rent falls short of her reservation rent. A government authority imposes a ceiling rent on every house (e.g., a legislated rent control). The central issue is *how* to efficiently allocate the houses among the tenants through a system of rents that satisfy both the reservation and ceiling rent constraints. Because price controls may fail the Walrasian equilibrium, the classical solution to the problem is to consider constrained equilibrium; see e.g., Andersson and Svensson (2014), Drèze (1975), or van der Laan (1980).

A major drawback of the constrained equilibrium is its lack of Pareto efficiency (Böhm and Müller, 1977; Herings and Konovalov, 2009). Therefore, we adopt the more natural solution of a core allocation. There are several major advantages. First, every core allocation is Pareto efficient and stable against any coalition deviation. Second, the core allocation is conceptually simpler, more intuitive and straightforward as it consists of only an assignment of items and its supporting price vector but does not use any rationing scheme. Third, the core allocation as a prime strategic equilibrium is widely used in general exchange economies and game theoretic models; see, e.g., Scarf (1967), Shapley and Scarf (1974), and Predtetchinski and Herings (2004).

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* Corresponding author.
E-mail addresses: tommy.andersson@nek.lu.se (T. Andersson), zaifu.yang@york.ac.uk (Z. Yang), d.zhang@uws.edu.au (D. Zhang).

1 For more on house allocation problems and matching markets, see, e.g., the surveys by Jackson (2013) and Serrano (2013).

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Our main result shows that the ascending auction of Talman and Yang (2008) can somewhat surprisingly always find a core allocation in finite steps. As a result, we provide an economically meaningful and effective solution to this class of practical regulated market problems.

2. The assignment model under price control

A seller wishes to sell n heterogeneous indivisible *items*, indexed by $1, \ldots, n$, to m *bidders* indexed by $1, \ldots, m$. The set $N = \{0, 1, \ldots, n\}$ represents all items including a *dummy item* 0, and the set $M = \{1, \ldots, m\}$ contains all bidders. The dummy item 0 has no value and can be assigned to any number of bidders. Every bidder $i \in M$ has a *valuation* $V^i(j) \in \mathbb{Z}_+$ in monetary units on each item $j \in N$ with $V^i(0) = 0$. The seller has a *reservation price* $\underline{p}_j \in \mathbb{Z}_+$ for each item $j \in N$ below which the item will not be sold. A central planner imposes *price controls*, i.e., each item $j \in N$ has a ceiling price $\overline{p}_j \in \mathbb{Z}_+$ above which the item is not permitted to sell. By convention $0 \le \underline{p}_j < \overline{p}_j$ for each $j \in N$ and $\underline{p}_0 = \overline{p}_0 = 0$. A *price vector* $p \in \mathbb{R}^N_+$ indicates a price for every item, and the set of *admissible prices* is given by

$$P = \{p \in \mathbb{R}^N_+ \mid p_0 = 0,\ \underline{p}_j \le p_j \le \overline{p}_j,\ j = 1, \ldots, n\}.$$

When there are no price controls, i.e., when $\overline{p}_j = +\infty$ for every $j = 1, \ldots, n$, the model reduces to the classical assignment market studied by, e.g., Koopmans and Beckmann (1957) and Shapley and Shubik (1971).

A *feasible assignment* π assigns every bidder $i \in M$ an item $\pi(i)$ such that no item in $N\backslash\{0\}$ is assigned to more than one bidder. Note that a feasible assignment may assign the dummy item to several bidders and a real item $j \neq 0$ may not be assigned to any bidder. An item $j \neq 0$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. Let $U(\pi)$ denote the set of all unassigned items at π. A *feasible allocation* (π, p) consists of a feasible assignment π and an admissible price vector p such that $p_j = \underline{p}_j$ for every unassigned item $j \in U(\pi)$. The *demand set* of bidder $i \in M$ at a price vector $p \in \mathbb{R}^N_+$ is given by

$$D^i(p) = \{j \in N \mid V^i(j) - p_j \ge V^i(k) - p_k \text{ for every } k \in N\}.$$

A *Walrasian equilibrium* is a feasible assignment π and a price vector $p \in \mathbb{R}^N_+$ such that $\pi(i) \in D^i(p)$ for all $i \in M$ and $p_j = \underline{p}_j$ for every unassigned item $j \in U(\pi)$. Price control may fail the Walrasian equilibrium since the equilibrium prices may not be admissible. To handle the problem, we will explore the notion of *core*.

Definition 2.1. A feasible allocation (π, p) is *a core allocation* if there does not exist a coalition S of bidders besides the seller and another feasible allocation (ρ, q) such that $\rho(i) = 0$ for every $i \in M\backslash S$, $V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)}$ for every $i \in S$, and $\sum_{h\in N} q_h > \sum_{h\in N} p_h$.

The following theorem establishes the existence of a core allocation, generalizing the well-known existence theorem for the assignment market by Koopmans and Beckmann (1957) and Shapley and Shubik (1971). It will be proved via an ascending auction in the next section.

Theorem 2.2. *The assignment model under price controls has at least one core allocation.*

3. Results

To present the auction we need some basic concepts. For a set of real items $T \subseteq N\backslash\{0\}$, and a vector $p \in \mathbb{R}^N_+$, define the *lower inverse demand* set of T at p by $D^-_T(p) = \{i \in M \mid D^i(p) \subseteq T\}$, i.e., the set of bidders who demand only items in T at p. The *upper inverse demand* of T at p is $D^+_T(p) = \{i \in M \mid D^i(p) \cap T \neq \emptyset\}$, i.e., the set of bidders who demand at least one item in T at p. A set of real items $O \subseteq N\backslash\{0\}$ is *overdemanded* at $p \in \mathbb{R}^N_+$ if $|D^-_O(p)| > |O|$, i.e., if the number of bidders who demand only items in O is strictly greater than the number of items in O. An overdemanded set O is *minimal* if no strict subset of O is overdemanded. A set of real items $T \subseteq N\backslash\{0\}$ is *underdemanded* at $p \in \mathbb{R}^N_+$ if $T \subseteq \{j \in N\backslash\{0\} \mid p_j > \underline{p}_j\}$ and $|D^+_T(p)| < |T|$, i.e., if the price of every item in T is strictly greater than its reservation price and the number of bidders who demand at least one item in T is strictly less than the number of items in T.

Talman and Yang (2008) proposed the following ascending auction for the assignment model under price control that finds a constrained equilibrium.

The ascending auction

Step 1: The auctioneer announces the set N of items, the reservation price vector $\underline{p}$ and ceiling prices $\overline{p}$. Set the iteration counter $t := 0$ and let $p^t := \underline{p}$, $N^0 := N$ and $M^0 := M$. Go to Step 2.

Step 2: The auctioneer asks each bidder $i \in M^t$ to report their demand sets $D^i(p^t)$ for the items in the set N^t and checks whether there is an overdemanded set in N^t at p^t. If there is no overdemanded set, the auction stops. Otherwise, there is at least one overdemanded set in N^t. The auctioneer first chooses a minimal overdemanded set $O \subseteq N^t$ and next checks whether the price of any item in O has reached its ceiling price. Let $\overline{O} := \{j \in O \mid p^t_j = \overline{p}_j\}$. If $\overline{O}$ is empty, the price of each item in O is increased by one unit and the prices of all other items are unchanged. Set $t := t + 1$ and return to Step 2. If $\overline{O}$ is nonempty, go to Step 3.

Step 3: The auctioneer picks an item j^* at random from $\overline{O}$ and asks all bidders $i \in M^t$ with $D^i(p^t) \subseteq O$ who demand the item to draw lots for the right to buy it. The (unique) winning bidder i^* gets item j^* by paying its current price and exits from the auction. Set $M^{t+1} = M^t\backslash\{i^*\}$ and $N^{t+1} = N^t\backslash\{j^*\}$. If $M^{t+1} = \emptyset$ or $N^{t+1} = \emptyset$, the auction stops. Otherwise, set $t := t + 1$ and return to Step 2.

If there is no price control, the above auction reduces to that of Demange et al. (1986). To prove the main result, we need to invoke two known results.

Lemma 3.1. *(van der Laan and Yang, 2008) Let O be minimal overdemanded at p. Then, $|D^+_T(p) \cap D^-_O(p)| > |T|$ for every nonempty subset T of O.*

Lemma 3.2. *(Mishra and Talman, 2010) There is a Walrasian equilibrium at $p \in \mathbb{R}^N_+$ if and only if there is neither an overdemanded set nor an underdemanded set at p.*

Lemma 3.3. *There is no underdemanded set in each step of the ascending auction.*

Proof. There is no underdemanded set at $p^0 = \underline{p} \in \mathbb{R}^N_+$ as the set $\{j \in N\backslash\{0\} \mid p_j > \underline{p}_j\}$ is empty. Suppose that there is no underdemanded set of items at p^t for $t > 0$, and, without loss of generality, that $p^t_j > \underline{p}_j$ for all $j \in N\backslash\{0\}$. We will prove that there is no underdemanded set of items at p^{t+1}. Let O be the chosen minimal overdemanded set at p^t. We need to consider two cases:

Case 1. Suppose that no item j in O has reached its ceiling price $\overline{p}_j$ at iteration t. Then, $p^{t+1}_k = p^t_k + 1$ for every $k \in O$ and $p^{t+1}_k = p^t_k$ for every $k \in N^t\backslash O$ by the rules of the auction. Because there is no underdemanded set at p^t, we have $|D^+_T(p^t)| \ge |T|$ for any set T of real items in N^t. For any $T \subseteq O$, by Lemma 3.1 the number of bidders in the lower inverse demand set $D^-_O(p^t)$ that demand at least one item of T at p^t is at least one more than the number of items in T. Since the valuations are integers and the increment for items in O is one, the number of bidders that demand at least one item of T at p^{t+1} will be the same as the number of bidders that demanded at least one of T at p^t. In all other cases, for any $T \subseteq N^t \setminus \{0\}$, the number of bidders that demand at least one item of T at p^{t+1} will be at least as big as the number of bidders that demanded at least one item of T at p^t. Hence, there is no underdemanded set at p^{t+1}.

Case 2. Suppose that some item j^* in O has reached its ceiling price $\overline{p}_j$. Then by the rules of the auction, the winning bidder $i^* \in M^t$ with $j^* \in D^{i^*}(p^t) \subseteq O$ leaves the auction with the item j^* by paying $p^t_{j^*} = \overline{p}_{j^*}$. As $p^{t+1}_j = p^t_j$ for all items in N^{t+1} by the rules of the auction, we can draw three conclusions. First, for any bidder $i \in M^{t+1}$ with $j^* \notin D^i(p^t)$, it holds that $D^i(p^{t+1}) = D^i(p^t)$. Second, for any bidder $i \in M^{t+1}$ with $j^* \in D^i(p^t)$ and $|D^i(p^t)| > 1$, it holds that $D^i(p^{t+1}) = D^i(p^t) \setminus \{j^*\}$. Third, for any bidder $i \in M^{t+1}$ with $D^i(p^t) = \{j^*\}$, it holds that $D^i(p^{t+1}) \subseteq N^t \setminus \{j^*\}$ (there can be at most one such bidder by Lemma 3.1 and the definition of a minimal overdemanded set). As the demand set of bidder $D^i(p^t)$ is a subset of O by the rules of the auction, it follows from the above three conclusions that if a subset T of N^{t+1} is underdemanded at p^{t+1}, then T must be a subset of $O \setminus \{j^*\}$ where $i^* \in D^+_T(p^t)$. Let $O' = O \setminus \{j^*\}$ and consider an arbitrary subset T of O' at p^{t+1} where $i^* \in D^+_T(p^t)$. Note that bidder i^* does not belong to M^{t+1} by the rules of the auction. From the above three conclusions it is then clear that all bidders except i^* that demand some item in T at p^t also demand some item from T at p^{t+1}. Hence

$$D^+_T(p^{t+1}) \cap D^-_{O'}(p^{t+1}) = (D^+_T(p^t) \cap D^-_O(p^t)) \setminus \{i^*\}.$$

Since T is a subset of O at p^t, it follows from Lemma 3.1 that $|D^+_T(p^t) \cap D^-_O(p^t)| > |T|$. The above observations and the fact that $|D^+_T(p^{t+1})| \geq |D^+_T(p^{t+1}) \cap D^-_{O'}(p^{t+1})|$ give

$$|D^+_T(p^{t+1})| \geq |D^+_T(p^{t+1}) \cap D^-_{O'}(p^{t+1})|$$
$$= |D^+_T(p^t) \cap D^-_O(p^t)| - 1 > |T| - 1.$$

Hence, $|D^+_T(p^{t+1})| \geq |T|$. But then T cannot be underdemanded at p^{t+1}. $\square$

The next theorem establishes that the allocation found by the ascending auction is always in the core, thus resulting in a constructive proof for Theorem 2.2.

Theorem 3.4. *The ascending auction always finds a core allocation in finite steps.*

Proof. That the auction terminates in finite steps follows immediately from the fact that the auction is ascending and that the valuations are finite. Let t^* be the iteration when the auction terminates. Because there is no underdemanded set and no overdemanded set at t^*, by construction and Lemma 3.3, there is a Walrasian equilibrium (π^*, p^{t^*}) for the bidders in M^{t^*} and items in N^{t^*} by Lemma 3.2. Together with their winning items and those bidders who paid ceiling prices before iteration t^*, we obtain a feasible allocation (π, p).

It remains to prove that (π, p) is a core allocation. Suppose not. Then there exists a coalition S of bidders besides the seller and an allocation (ρ, q) blocking (π, p). For the seller, we have

$$\sum_{j \in N} q_j = \sum_{i \in S} q_{\rho(i)} + \sum_{j \in U(\rho)} \underline{p}_j > \sum_{i \in M} p_{\pi(i)} + \sum_{j \in U(\pi)} \underline{p}_j = \sum_{j \in N} p_j.$$

It is clear that there exists some $j^* \in N$ such that

$$q_{j^*} > p_{j^*}. \tag{3.1}$$

This means that some bidder $i^* \in S$ must be assigned item j^* at ρ, i.e., $\rho(i^*) = j^*$. On the other hand, for every bidder $i \in S$, we have

$$V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)}. \tag{3.2}$$

For bidder i^*, it follows from conditions (3.1) and (3.2) that

$$V^{i^*}(j^*) - p_{j^*} > V^{i^*}(j^*) - q_{j^*} > V^{i^*}(\pi(i^*)) - p_{\pi(i^*)}.$$

But this inequality implies that at p, bidder i^* should have rejected item $\pi(i^*)$ in favor of item j^*, yielding a contradiction. This argument is valid, because item j^* cannot be an item that has reached its upper price level $\overline{p}_{j^*}$ and has been sold before the auction stops, for otherwise, we would have $q_{j^*} = p_{j^*} = \overline{p}_{j^*}$ which contradicts inequality (3.1). This demonstrates that (π, p) is a core allocation. $\square$

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