# On Axiomatization of Epistemic GDL ⋆

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**Abstract.** The Game Description Language (GDL) has been introduced as an official language for specifying games in the AAAI General Game Playing Competition since 2005. It was originally designed as a declarative language for representing rules of arbitrary games with perfect information. More recently, an epistemic extension of GDL, called EGDL, has been proposed for representing and reasoning about imperfect information games. In this paper, we develop an axiomatic system for a variant of EGDL and prove its soundness and completeness with respect to the semantics based on the epistemic state transition model. With a combination of action symbols, temporal modalities and epistemic operators, the completeness proof requires novel combinations of techniques used for completeness of propositional dynamic logic and epistemic temporal logic. We demonstrate how to use the proof theory for inferring game properties from game rules.

## 1 Introduction

General Game Playing (GGP) is concerned with creating intelligent agents that can play previously unknown games by just being given their rules [6]. To specify a game played by autonomous agents, a formal game description language, called GDL, has been introduced as an official language for GGP since 2005. GDL is defined as a high-level, machine-processable language for representing the rules of arbitrary games with perfect information [16]. Originally designed as a logic programming language, GDL has been recently adapted as a logical language for game specification and strategic reasoning [25]. Based on this, the epistemic extension of GDL, called EGDL, has been developed for representing and reasoning about imperfect information games [14].

Syntactically, EGDL extends GDL with the standard epistemic operators to specify the rules of imperfect information games and capture the epistemic status of agents. For example, an EGDL-formula $\mathsf{K}_r(does(a) \wedge \bigcirc wins(r)) \rightarrow does(a)$ specifies that if an agent knows that taking an action leads to win at the next state, then she takes that action at the current state. Semantically, EGDL is interpreted over epistemic state transition models which are used to represent synchronous and deterministic games with imperfect information. The expressive power and computational efficiency of EGDL have been investigated in [14]. In this paper, we address the fundamental logical question of a complete axiomatization for EGDL.

⋆ Most of the work was done while the first author was a postdoc at IRIT, University of Toulouse.

The axiomatic system for EGDL that we present is composed of axiom schemes and inference rules that capture the logical properties of the semantical models. With action, temporal and epistemic operators, the completeness proof of EGDL, however, is non-trivial. It requires novel combinations and extensions of techniques from both propositional dynamic logic (PDL) [15] and epistemic temporal logics (ETLs) [7].

To achieve the completeness of EGDL, we first construct a pre-model for a consistent EGDL-formula $\varphi$ out of maximal consistent subsets of a finite set of formulas, called the *closure* of $\varphi$. Similar to [7], we define a number of different distinct levels of closure so as to deal with the epistemic operators. The techniques to construct the pre-model has been strongly influenced by two sources which gave us valuable insights: [15] providing an elementary proof of the completeness of PDL, and [7] presenting a general framework for completeness proofs of ETLs. Unfortunately, the pre-model is non-deterministic and thus is not an epistemic state transition model. To fill this gap, we then transform the pre-model into an epistemic state transition model with an equivalent satisfiability of EGDL-formulas. Such transformation is inspired by the method used in [18] to transform a non-deterministic automata into a deterministic one. From the completeness proof, we derive the *finite model property* for EGDL: every EGDL-formula that is satisfiable in some epistemic state transition model is satisfiable in a finite epistemic state transition model. We also demonstrate how to use the proof theory for inferring game properties from game rules.

The rest of this paper is structured as follows: Section 2 establishes the syntax and the semantics of EGDL. Section 3 provides a sound and complete axiomatic system for EGDL and demonstrates how to use the proof theory for reasoning about game rules. Section 4 discusses the related work. Finally we conclude with future work.

## 2 The Framework

All games are assumed to be played in multi-agent environments. Each game is associated with a game signature. A *game signature* $\mathcal{S}$ is a triple $(N, \mathcal{A}, \Phi)$, where $N = \{1, 2, \cdots, m\}$ is a non-empty finite set of agents; $\mathcal{A} = \bigcup_{r \in N} A^r$, where $A^r$ consists of a non-empty finite set of *actions* for agent $r$ such that different agents have different actions, i.e., $A^{r_1} \cap A^{r_2} = \emptyset$ if $r_1 \neq r_2$ and each agent has an action without effects, i.e., $noop^r \in A^r$, and $\Phi = \{p, q, \cdots\}$ is a finite set of propositional atoms for specifying individual features of a game state.

Through the rest of the paper, we will fix a game signature $\mathcal{S}$ and all concepts will be based on this game signature unless otherwise specified.

### 2.1 Epistemic State Transition Models

We consider synchronous imperfect information games where all players move simultaneously and may have partial information of the game states. The structures of these games may be specified by epistemic state transition models defined as follows:

**Definition 1.** *An* epistemic state transition (EST) model $M$ *is a tuple* $(W, I, T, \{R_r\}_{r \in N}, \{L_r\}_{r \in N}, U, g, \pi)$, *where*

- $W$ *is a non-empty set of* possible states*.*
- $I \subseteq W$*, representing a set of* initial *states.*
- $T \subseteq W \backslash I$*, representing a set of* terminal *states.*
- $R_r \subseteq W \times W$ *is an equivalence relation for agent* $r$*, indicating the states that are indistinguishable for* $r$*.*
- $L_r \subseteq W \times A^r$ *is a* legality *relation for agent* $r$*, describing the legal actions of agent* $r$ *at each state. Let* $L_r(w) = \{a \in A^r : (w, a) \in L_r\}$ *be the set of all legal actions of agent* $r$ *at state* $w$*. To make a game playable, we assume that (i) each agent has at least one available action at each state:* $L_r(w) \neq \emptyset$ *for all* $r \in N$ *and* $w \in W$*, and (ii) at all terminal states each agent can only take action* $noop$*:* $L_r(w) = \{noop^r\}$ *for any* $r \in N$ *and* $w \in T$*.*
- $U : W \times \prod_{r \in N} A^r \hookrightarrow W \backslash I$ *is a partial* update *function, specifying the state transformations, such that* $U(w, \langle noop^r \rangle_{r \in N}) = w$ *for any* $w \in T$*.*
- $g : N \rightarrow 2^W$ *is a* goal *function, specifying the winning states for each agent.*
- $\pi : W \rightarrow 2^\Phi$ *is a standard valuation function.*

Note that different from [14], (i) we consider a general case without assuming a unique initial state; (ii) the update function is partial, as not all joint actions are possible in all states due to the legality relation. In particular, there is no semantical condition to guarantee that all joint legal actions lead to valid next states. Such a condition is not easy to provide, giving the legal conditions are defined for individual agents. Besides this, we do not require each agent knows her own legal actions, since in GGP it may occur that an agent fails to figure out her legal actions given the limited time. In that case, the game master assigns a random legal action for her. For convenience, let $D$ denote the set of all joint actions $\prod_{r \in N} A^r$. For $d \in D$, let $d(r)$ denote agent $r$'s action in the joint action $d$. We write $R_r(w)$ for the set of all states that agent $r$ cannot distinguish from $w$, i.e., $R_r(w) = \{u \in W \ : \ wR_r u\}$. We now define the notion of *a path* to specify the set of all possible ways in which a game can develop.

**Definition 2.** *Given an EST-model* $M = (W, I, T, \{R_r\}_{r \in N}, \{L_r\}_{r \in N}, U, g, \pi)$*, a path* $\delta$ *is an infinite sequence of states and actions* $w_0 \xrightarrow{d_1} w_1 \xrightarrow{d_2} w_2 \cdots \xrightarrow{d_j} \cdots$ *such that for all* $j \geq 1$ *and for any* $r \in N$*,*

1. $w_j = U(w_{j-1}, d_j)$ *(state update);*
2. $(w_{j-1}, d_j(r)) \in L_r$ *(that is, any action that is taken must be legal.);*
3. *if* $w_{j-1} \in T$*, then* $w_{j-1} = w_j$ *(that is, a loop after reaching a terminal state.).*

It follows that only the first state may be initial, i.e., $w_j \notin I$. Let $\mathcal{P}(M)$ denote the set of all paths in $M$. When $M$ is fixed, we simply write $\mathcal{P}$. For a path $\delta \in \mathcal{P}$ and a position $j \geq 0$, we use $\delta[j]$, $\delta[0, j]$ and $\delta[j, \infty]$ to denote the $j$-th state of $\delta$, the finite prefix $w_0 \xrightarrow{d_1} w_1 \xrightarrow{d_2} \cdots \xrightarrow{d_j} w_j$ of $\delta$ and the infinite suffix path $w_j \xrightarrow{d_{j+1}} w_{j+1} \xrightarrow{d_{j+2}} \cdots$ of $\delta$, respectively. Finally, we write $\theta_r(\delta, j)$ for the action of agent $r$ taken at stage $j$ of $\delta$.

The following definition, by extending equivalence relations over states to paths, characterizes precisely what an agent with *imperfect recall and perfect reasoning* can in principle know during a game.

**Definition 3.** *Two paths* $\delta, \delta' \in \mathcal{P}$ *are imperfect recall (also called memoryless) equivalent for agent* $r$*, written* $\delta \approx_r \delta'$*, iff* $\delta[0] R_r \delta'[0]$*.*

That is, imperfect recall requires an agent to be only aware of the present state but forget everything that happened. This is similar to the notion of imperfect recall in ATL [20].

### 2.2 The Syntax

Let us now introduce an epistemic extension of the game description language GDL [25] to represent games with imperfect information. We further provide a semantics for the language based on the epistemic state transition model. In the following, we call this resulting framework EGDL for short.

**Definition 4.** *The language $\mathcal{L}$ of EGDL is generated by the following BNF:*

$$\varphi ::= p \mid initial \mid terminal \mid legal(a^r) \mid wins(r) \mid does(a^r) \mid$$

$$\neg\varphi \mid \varphi \wedge \psi \mid \bigcirc\varphi \mid \mathsf{K}_r\varphi \mid \mathsf{C}\varphi$$

*where $p \in \Phi$, $r \in N$ and $a^r \in A^r$.*

Other connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$ are defined by $\neg$ and $\wedge$ in the standard way. Intuitively, $initial$ and $terminal$ specify the initial state and the terminal states of a game, respectively; $does(a^r)$ asserts that agent $r$ takes action $a$ at the current state; $legal(a^r)$ asserts that action $a$ is available to agent $r$ at the current state; and $wins(r)$ asserts that agent $r$ wins at the current state. The formula $\bigcirc\varphi$ means "$\varphi$ holds in the next state". All these components are inherited from GDL. The epistemic operators $\mathsf{K}$ and $\mathsf{C}$ are taken from the Modal Epistemic Logic [9, 5]. The formula $\mathsf{K}_r\varphi$ is read as "agent $r$ knows $\varphi$", and $\mathsf{C}\varphi$ as "$\varphi$ is common knowledge among all the agents in $N$". As usual, we write $\widehat{\mathsf{K}}_r$ for the dual of $\mathsf{K}_r$ and $\mathsf{E}\varphi$ for $\bigwedge_{r \in N} \mathsf{K}_r\varphi$, saying that every agent in $N$ knows $\varphi$.

To illustrate the intuition of the language, let us consider a variant of the Tic-Tac-Toe, called Krieg-Tictactoe in [19].

*Example 1.* Krieg-Tictactoe is played by two players, cross $\mathsf{x}$ and naught $\mathsf{o}$, who take turns marking cells in a $3 \times 3$ board. Different from standard Tic-Tac-Toe, each player can see her own marks, but not those of her opponent, just like the chess variant *Kriegspiel* [17].

To represent the Krieg-Tictactoe, we first describe its game signature, written $\mathcal{S}_{KT}$, as follows: $N_{KT} = \{\mathsf{x}, \mathsf{o}\}$ where $\mathsf{x}$ denotes the player who marks the symbol cross and $\mathsf{o}$ denotes the player who marks the symbol naught; $A^r_{KT} = \{a^r_{i,j} : 1 \leq i, j \leq 3\} \cup \{noop^r\}$, where $a^r_{i,j}$ denotes the action that player $r$ marks cell $(i, j)$ with her symbol; $\Phi_{KT} = \{p^r_{i,j}, tried(a^r_{i,j}), turn(r) : r \in \{\mathsf{x}, \mathsf{o}\}$ and $1 \leq i, j \leq 3\}$, where $p^r_{i,j}$ represents the fact that cell $(i, j)$ is marked with player $i$'s symbol, $tried(a^r_{i,j})$ represents the fact that player $r$ has tried to mark cell $(i, j)$ but failed before, and $turn(r)$ says that it is player $r$'s turn now. The rules of Krieg-Tictactoe are specified by EGDL in Figure 1 (where $1 \leq i, j \leq 3$, $r \in \{\mathsf{x}, \mathsf{o}\}$ and $-r$ represents $r$'s opponent).

Rules 1-5 specify the initial state, each player's winning states, the terminal states and the turn-taking. In particular, Rule 2 specifies that if a player has tried to mark a cell, then the corresponding cell is marked by the opponent.

1. $initial \leftrightarrow turn(\mathsf{x}) \wedge \neg turn(\mathsf{o}) \wedge \bigwedge_{i,j=1}^{3} \neg(p^{\mathsf{x}}_{i,j} \vee p^{\mathsf{o}}_{i,j})$
2. $tried(a^r_{i,j}) \rightarrow p^{-r}_{i,j}$
3. $wins(r) \leftrightarrow (\bigvee_{i=1}^{3} \bigwedge_{l=0}^{2} p^r_{i,1+l}) \vee (\bigvee_{j=1}^{3} \bigwedge_{l=0}^{2} p^r_{1+l,j}) \vee (\bigwedge_{l=0}^{2} p^r_{1+l,1+l}) \vee (\bigwedge_{l=0}^{2} p^r_{1+l,3-l})$
4. $teminal \leftrightarrow wins(\mathsf{x}) \vee wins(\mathsf{o}) \vee \bigwedge_{i,j=1}^{3} (p^{\mathsf{x}}_{i,j} \vee p^{\mathsf{o}}_{i,j})$
5. $turn(r) \wedge \neg terminal \rightarrow \bigcirc\neg turn(r) \wedge \bigcirc turn(-r)$
6. $legal(noop^r) \leftrightarrow turn(-r) \vee terminal$
7. $legal(a^r_{i,j}) \leftrightarrow turn(r) \wedge \neg p^r_{i,j} \wedge \neg tried(a^r_{i,j}) \wedge \neg terminal$
8. $\bigcirc p^r_{i,j} \leftrightarrow p^r_{i,j} \vee (does(a^r_{i,j}) \wedge \neg(p^{\mathsf{x}}_{i,j} \vee p^{\mathsf{o}}_{i,j}))$
9. $\bigcirc tried(a^r_{i,j}) \leftrightarrow tried(a^r_{i,j}) \vee (does(a^r_{i,j}) \wedge p^{-r}_{i,j})$
10. $does(a^r_{i,j}) \rightarrow \mathsf{K}_r(does(a^r_{i,j}))$
11. $initial \rightarrow \mathsf{E} initial$
12. $(turn(r) \rightarrow \mathsf{E} turn(r)) \wedge (\neg turn(r) \rightarrow \mathsf{E}\neg turn(r))$
13. $(p^r_{i,j} \rightarrow \mathsf{K}_r p^r_{i,j}) \wedge (\neg p^r_{i,j} \rightarrow \mathsf{K}_r \neg p^r_{i,j})$
14. $(tried(a^r_{i,j}) \rightarrow \mathsf{K}_r tried(a^r_{i,j})) \wedge (\neg tried(a^r_{i,j}) \rightarrow \mathsf{K}_r \neg tried(a^r_{i,j}))$

**Fig. 1.** An EGDL description of Krieg-Tictactoe.

The preconditions of each action (legality) are specified by 6 and 7. *The player who has the turn can mark any non-terminal cell such that (i) it is not marked by herself, and (ii) she has never tried to mark it before. A player can only do action $noop$ at the terminal states or the states where it is not her turn.*

Rules 8 and 9 are the combination of the frame axioms and the effect axioms. Rule 8 states that *a cell is marked with a player's symbol in the next state if the player takes the corresponding action at the current state or the cell has been marked by her symbol before.* Similarly, Rule 9 says that *an action is tried by a player in the next state if the action is ineffective while still taken by the player at the current state, or this action has been tried before.*

The rest of the rules specify the epistemic status of the game. Rule 10 states each player knows which action she is taking. Rule 11 and Rule 12 say both players know the initial state and their turns, respectively. Rule 13 says that *each player knows which cell is marked or not with her symbol.* Similarly, Rule 14 states that *each player knows which cell is tried or not by herself.*

Note that rules 12-14 together specify the epistemic relations for each player: *two states are indistinguishable for a player if their configurations are the same from her point of view*. Finally, let $\Sigma_{KT}$ be the set of rules 1-14.

### 2.3 The Semantics

The semantics of EGDL-formulas is based on the epistemic state transition models.

**Definition 5.** *Let $M$ be an EST-model. Given a path $\delta$ in $M$ and a formula $\varphi \in \mathcal{L}$, we say $\varphi$ is true at $\delta$ under $M$, denoted by $M, \delta \models \varphi$, according to the following definition*:

$M, \delta \models p$ *iff* $p \in \pi(\delta[0])$

$$\begin{array}{lll}
M,\delta \models \neg\varphi & \textit{iff} & M,\delta \not\models \varphi \\
M,\delta \models \varphi_1 \wedge \varphi_2 & \textit{iff} & M,\delta \models \varphi_1 \textit{ and } M,\delta \models \varphi_2 \\
M,\delta \models initial & \textit{iff} & \delta[0] \in I \\
M,\delta \models terminal & \textit{iff} & \delta[0] \in T \\
M,\delta \models wins(r) & \textit{iff} & \delta[0] \in g(r) \\
M,\delta \models legal(a^r) & \textit{iff} & (\delta[0], a^r) \in L_r \\
M,\delta \models does(a^r) & \textit{iff} & \theta_r(\delta, 0) = a^r \\
M,\delta \models \bigcirc\varphi & \textit{iff} & M,\delta[1,\infty] \models \varphi \\
M,\delta \models \mathsf{K}_r\varphi & \textit{iff} & \textit{for any } \delta' \in \mathcal{P}, \textit{ if } \delta \approx_r \delta', \textit{ then } M,\delta' \models \varphi \\
M,\delta \models \mathsf{C}\varphi & \textit{iff} & \textit{for any } \delta' \in \mathcal{P}, \textit{ if } \delta \approx_N \delta', \textit{ then } M,\delta' \models \varphi
\end{array}$$

*where* $\approx_N$ *is the transitive closure of* $\bigcup_{r\in N} \approx_r$.

A formula $\varphi$ is *globally true or valid* in an EST-model $M$, written $M \models \varphi$, if $M,\delta \models \varphi$ for any $\delta \in \mathcal{P}$. A formula $\varphi$ is *valid*, written $\models \varphi$, if $M \models \varphi$ for any EST-model $M$. Let $\Sigma$ be a set of formulas in $\mathcal{L}$, then $M$ is a *model* of $\Sigma$ if $M \models \varphi$ for all $\varphi \in \Sigma$.

The following result specifies some generic game properties.

**Proposition 1.** *For any* $r \in N$, $\varphi \in \mathcal{L}$ *and* $a^r, b^r \in A^r$,

1. $\models \neg \bigcirc initial$
2. $\models terminal \rightarrow \bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg legal(a^r) \wedge legal(noop^r)$
3. $\models \bigvee_{a^r \in A^r} does(a^r)$
4. $\models \neg(does(a^r) \wedge does(b^r))$ *for* $a^r \neq b^r$
5. $\models does(a^r) \rightarrow legal(a^r)$
6. $\models \bigvee_{a^r \in A^r} legal(a^r)$
7. $\models terminal \wedge \varphi \rightarrow \bigcirc\varphi$

The first formula says that a game would never go back to its initial state once it starts. The second formula specifies that all players can only take action “noop” at the terminal states. The third and forth formulas prescribe that there is a unique action for each player at all game states. The fifth formula asserts that any action that is taken should be legal. The sixth formula specifies that each player has at least one legal action at each state. The last formula requires that a terminal state leads to a self-loop.

Besides those generic game properties, EGDL is also able to specify epistemic properties of a game. For instance, whether each player always knows her own legal actions in the course of the game. This property as well as some other well-known properties have been discussed in [14]. It is worth of mentioning that although these properties are expressible in EGDL, different from the generic game properties, they are not valid for any game model.

## 3 Axiomatization

In this section, we develop an axiomatic system for the logic EGDL, and provide its soundness and completeness with respect to the epistemic state transition models.

### 3.1 The Axiomatic System

EGDL consists of the following axiom schemas and inference rules: For any $a^r, b^r \in A^r$, $r \in N$ and $\varphi, \psi \in \mathcal{L}$,

- Axiom Schemas:
  1. All tautologies of classical propositional logic.
  2. $\neg \bigcirc initial$
  3. $terminal \rightarrow \bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg legal(a^r) \wedge legal(noop^r)$
  4. $\bigvee_{a^r \in A^r} does(a^r)$
  5. $\neg(does(a^r) \wedge does(b^r))$ for $a^r \neq b^r$.
  6. $\bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi)$
  7. $\neg \bigcirc \varphi \leftrightarrow \bigcirc \neg\varphi$
  8. $does(a^r) \rightarrow legal(a^r)$
  9. $\varphi \wedge terminal \rightarrow \bigcirc\varphi$
  10. $\mathsf{K}_r(\varphi \rightarrow \psi) \rightarrow (\mathsf{K}_r\varphi \rightarrow \mathsf{K}_r\psi)$
  11. $\mathsf{K}_r\varphi \rightarrow \varphi$
  12. $\mathsf{K}_r\varphi \rightarrow \mathsf{K}_r\mathsf{K}_r\varphi$
  13. $\neg\mathsf{K}_r\varphi \rightarrow \mathsf{K}_r\neg\mathsf{K}_r\varphi$
  14. $\mathsf{E}\varphi \leftrightarrow \bigwedge_{r=1}^{m} \mathsf{K}_r\varphi$
  15. $\mathsf{C}\varphi \rightarrow \mathsf{E}(\varphi \wedge \mathsf{C}\varphi)$
- Inference Rules:
  (R1) From $\varphi$, $\varphi \rightarrow \psi$ infer $\psi$.
  (R2) From $\varphi$ infer $\bigcirc\varphi$.
  (R3) From $\varphi$ infer $\mathsf{K}_r\varphi$.
  (R4) From $\varphi \rightarrow \mathsf{E}(\varphi \wedge \psi)$ infer $\varphi \rightarrow \mathsf{C}\psi$.

Besides the axioms mentioned in Proposition 1, the axioms for temporal and epistemic operators are well-known. Note that since we focus on games with imperfect recall, thus there is no interaction properties between epistemic and temporal operators. Let $\vdash$ denote the provability in EGDL. The notion of *the syntactic consequence (derivation)* is defined in the standard way.

With the proof theory, we are now able to derive the following formulas from the rules of Krieg-Tictactoe specified in Figure 1.

**Proposition 2.** *For any $r \in N_{KT}$ and $a^r_{i,j} \in A^r_{KT}$,*

1. $\vdash_{\Sigma_{KT}} initial \rightarrow \mathsf{C}initial$
2. $\vdash_{\Sigma_{KT}} legal(a^r_{i,j}) \rightarrow \mathsf{K}_r(legal(a^r_{i,j}))$
3. $\vdash_{\Sigma_{KT}} does(a^r_{i,j}) \rightarrow \bigcirc\mathsf{K}_r(p^r_{i,j} \vee tried(a^r_{i,j}))$
4. $\vdash_{\Sigma_{KT}} \mathsf{K}_r tried(a^r_{i,j}) \rightarrow \mathsf{K}_r p^{-r}_{i,j}$

That is, in Krieg-Tictactoe, the turn-taking is common knowledge (Clause 1). Each player knows her own available actions (Clause 2). If an agent marks a cell at the current state, then she will knows either this cell has been marked or been tried by herself at the next state (Clause 3). Moreover, if a player knows that she has tried to mark a cell, then she knows the corresponding cell has been marked by the opponent (Clause 4).

### 3.2 Completeness Proofs

The completeness result is achieved in two step. First, we construct a pre-model for a consistent formula $\varphi$ out of consistent subsets of a finite set of formulas, called the *closure of* $\varphi$. The construction resembles those previously used for completeness of propositional dynamic logic [15] and epistemic temporal logics [7]. Next we transform the pre-model into an epistemic state transition model and show that the satisfiability of EGDL-formulas is invariant under such transformation. This idea is captured in Figure 2 (where $r \in N$ and $k \in \mathbb{N}$).

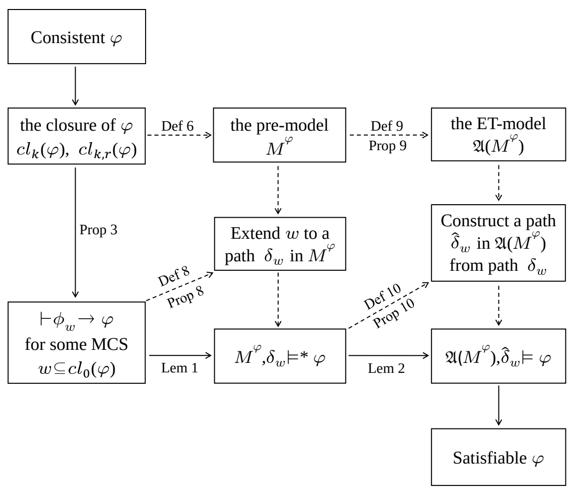


**Fig. 2.** The roadmap of the completeness proof for EGDL. Note that solid arrows denote the process to achieve the completeness, and dashed arrows denote the notions and their properties to obtain the intermediate results. The abbreviation "MCS" denotes the maximal consistent set.

Let us now fix a formula $\varphi \in \mathcal{L}$, which is consistent in EGDL, i.e., not $\vdash \neg\varphi$. We define $ad(\varphi)$ to be the greatest number of alternations of distinct $\mathsf{K}_r$'s along any branch in $\varphi$'s parse tree. If $\varphi$ involves the common knowledge operator $\mathsf{C}$, let $ad(\varphi) = 0$. For instance, $ad(\mathsf{K}_{r_1}\mathsf{K}_{r_2}\mathsf{K}_{r_1}p) = 3$; $ad(\mathsf{K}_{r_1}\mathsf{K}_{r_1}\mathsf{K}_{r_2}p) = 2$; $ad(\mathsf{C}\mathsf{K}_{r_1}\mathsf{K}_{r_2}p) = 0$; temporal operators are not considered, so that $ad(\mathsf{K}_{r_1}\mathsf{K}_{r_2} \bigcirc \mathsf{K}_{r_1}p) = 3$.

A finite sequence $\sigma = r_1 r_2 \cdots r_k$ of agents, possibly equal to the null sequence $\epsilon$, is called an *index* if $r_i \neq r_{i+1}$ for all $i < k$. We write $|\sigma|$ for the length $k$ of such a sequence. In particular, $|\epsilon| = 0$.

Let $N^*$ be the set of all finite sequences over $N$, we define the absorptive concatenation function $\#$ from $N^* \times N$ to $N^*$ as follows: Given a sequence $\sigma \in N^*$ and an agent $r \in N$,

$$\sigma\#r = \begin{cases} \sigma & \text{if the final element of } \sigma \text{ is } r; \\ \sigma r & \text{otherwise.} \end{cases}$$

Given $\varphi \in \mathcal{L}$, for each $k \geq 0$, we define the $k$-closure $cl_k(\varphi)$, and for each agent $r \in N$, we define the $k,r$-closure $cl_{k,r}(\varphi)$. The definitions of these sets proceeds by mutual recursion:

1. *The basic closure* $cl_0(\varphi)$ is the smallest set containing $\varphi$ such that
   (a) it is closed under subformulas.
   (b) if $\mathsf{E}\psi \in cl_0(\varphi)$, then $\mathsf{K}_{r_1}\psi, \cdots, \mathsf{K}_{r_m}\psi \in cl_0(\varphi)$.
   (c) if $\mathsf{C}\psi \in cl_0(\varphi)$, then $\mathsf{EC}\psi \in cl_0(\varphi)$.
   (d) if $\psi \in cl_0(\varphi)$ and $\psi$ is not of the form $\neg\psi'$, then $\neg\psi \in cl_0(\varphi)$.
2. Let $cl_{k,r}(\varphi)$ be the union of $cl_k(\varphi)$ with the set of formulas of the form $\mathsf{K}_r(\psi_1 \vee \cdots \psi_n)$ or $\neg\mathsf{K}_r(\psi_1 \vee \cdots \psi_n)$, where the $\psi_i$ are distinct formulas in $cl_k(\varphi)$.
3. $cl_{k+1}(\varphi) = \bigcup_{r=1}^{m} cl_{k,r}(\varphi)$.

If $X$ is a finite set of formulas, we write $\phi_X$ for the conjunction of all the formulas in $X$. A finite set $X$ of formulas is said to be consistent if $\phi_X$ is consistent. A finite set $Cl$ of formulas is said to be *negation-closed* if, for all $\psi \in Cl$, either $\neg\psi \in Cl$ or $\psi$ is of the form $\neg\psi'$ and $\psi' \in Cl$. Note that the sets $cl_k(\varphi)$ and $cl_{k,r}(\varphi)$ are negation-closed. We define an *atom* of $Cl$ to be a maximal consistent subset of $Cl$. The set of all atoms of $Cl$ is denoted as $\mathcal{A}_{Cl}$. We have the following properties.

**Proposition 3.** *Suppose that $X$ is a finite set of formulas and $Cl$ is a negation-closed set of formulas. For any $\varphi_1, \varphi_2 \in \mathcal{L}$,*

1. *if $\vdash \phi_X \to \varphi_1$ and $\vdash \varphi_1 \to \varphi_2$, then $\vdash \phi_X \to \varphi_2$.*
2. *if $X$ is an atom of $Cl$ and $\psi \in Cl$, then either $\vdash \phi_X \to \psi$ or $\vdash \phi_X \to \neg\psi$.*
3. $\vdash \bigvee_{X \in \mathcal{A}_{Cl}} \phi_X$.

The construction of the pre-model of $\varphi$ is based on the atoms of the closures of $\varphi$. Let $d = ad(\varphi)$.

**Definition 6.** *The pre-model of $\varphi$, denoted by $M^\varphi = (W^\varphi, I^\varphi, T^\varphi, \{R_r^\varphi\}_{r\in N}, \{L_r^\varphi\}_{r\in N}, U^\varphi, g^\varphi, \pi^\varphi)$, is constructed as follows:*

1. *$W^\varphi$ consists of all the pairs $(\sigma, X)$ such that $\sigma$ is an index, $|\sigma| \le d$, and*
   *(a) if $\sigma = \epsilon$ then $X$ is an atom of $cl_d(\varphi)$, and*
   *(b) if $\sigma = \tau r$ then $X$ is an atom of $cl_{k,r}(\varphi)$, where $k = d - |\sigma|$.*
2. $I^\varphi = \{(\sigma, X) \in W^\varphi \;:\; \vdash \phi_X \to initial\}$.
3. $T^\varphi = \{(\sigma, X) \in W^\varphi \;:\; \vdash \phi_X \to terminal\}$.
4. *$(\sigma, X) R_r^\varphi (\tau, Y)$ iff $\sigma \# r = \tau \# r$ and $\{\psi : \mathsf{K}_r\psi \in X\} = \{\chi : \mathsf{K}_r\chi \in Y\}$.*
5. *$((\sigma, X), a^r) \in L_r^\varphi$ iff $\vdash \phi_X \to legal(a^r)$.*
6. *$U^\varphi((\sigma, X), d) = (\tau, Y)$ iff $\vdash \phi_X \to \bigwedge_{r\in N} legal(d(r))$, $\sigma = \tau$ and the formula $\phi_X \wedge \bigcirc\phi_Y$ is consistent.*
7. $g^\varphi(r) = \{(\sigma, X) \in W^\varphi \;:\; \vdash \phi_X \to wins(r)\}$.
8. $\pi^\varphi((\sigma, X)) = \{p \in \Phi \;:\; \vdash \phi_X \to p\}$.

It is easy to see that the update function in $M^\varphi$ is non-deterministic, and thus $M^\varphi$ is not an EST-model. We redefine the notion of a path as follows:

**Definition 7.** *Given the pre-model $M^\varphi = (W^\varphi, I^\varphi, T^\varphi, \{R_r^\varphi\}_{r\in N}, \{L_r^\varphi\}_{r\in N}, U^\varphi, g^\varphi, \pi^\varphi)$ of $\varphi$, a path $\delta$ of $M^\varphi$ is an infinite sequence of states and actions $w_0 \xrightarrow{d_1} w_1 \xrightarrow{d_2} \cdots$ such that the conditions are the same as those in Definition 2 except changing Condition 1 to $w_j \in U^\varphi(w_{j-1}, d_j)$ due to nondeterminacy.*

Similarly, we generalize the indistinguishable relation to the paths. We say that two paths $\delta, \delta'$ of $M^\varphi$ are indistinguishable for agent $r$, denoted by $\delta \approx_r^\varphi \delta'$, iff $\delta[0] R_r^\varphi \delta'[0]$. The truth conditions for all EGDL-formulas under the pre-model are exactly the same as those in Definition 5. In particular, we use $M^\varphi, \delta \models^* \varphi$ to denote that $\varphi$ is true at path $\delta$ under $M^\varphi$.

If $w = (\sigma, X)$ is a state, we define $\phi_w$ to be the formula $\phi_X$. Following [7], we say that the state $w$ *directly decides* a formula $\psi$ if either $\psi \in w$, $\neg\psi \in w$, or $\psi = \neg\psi'$ and $\psi' \in w$. We say that $w$ *decides* $\psi$ if either $\vdash \phi_w \to \psi$, or $\vdash \phi_w \to \neg\psi$. Clearly, if $w$ directly decides $\psi$, then $w$ decides $\psi$. Note that if $\sigma = \tau r$, then each $\sigma$-state directly decides every formula in $cl_{k,r}(\varphi)$. Also, every $\epsilon$-state directly decides every formula in $cl_d(\varphi)$. In particular, we have the following results about formulas with K-operators.

**Proposition 4.** *Given two states $w = (\sigma, X)$ and $u = (\tau, Y)$, if $\sigma\#r = \tau\#r$, then the same formulas of the form $\mathsf{K}_r\psi$ are directly decided by $w$ and $u$.*

Given a $\sigma$-state $w$, we use $\Phi_{w,r}$ for the disjunction of all the formulas $\phi_u$, where $u$ is a $\sigma$-state satisfying $w R_r^\varphi u$, and we use $\Phi_{w,r}^+$ for the disjunction of all the formulas $\phi_u$, where $u$ is a $\sigma\#r$-state satisfying $w R_r^\varphi u$.

**Proposition 5.**

1. *If $w$ is a $\sigma$-state and $u$ is a $\sigma$-state or $\sigma\#r$-state such that not $w R_r^\varphi u$, then $\vdash \phi_w \to \mathsf{K}_r\neg\phi_u$.*
2. *For all $\sigma$-states $w$, $\vdash \phi_w \to \mathsf{K}_r\Phi_{w,r}$.*
3. *For all $\sigma$-states $w$, if $|\sigma\#r| \leq d$, then $\vdash \phi_w \to \mathsf{K}_r\Phi_{w,r}^+$.*

The following two propositions show that the pre-model has properties resembling those for the truth conditions for formulas in the basic closure.

**Proposition 6.** *For all $\sigma$-states $w$ and $\mathsf{K}_r\psi \in cl_0(\varphi)$, the following are equivalent.*

1. $\vdash \phi_w \to \neg\mathsf{K}_r\psi$.
2. *There is some $\sigma$-state $u$ such that $w R_r^\varphi u$ and $\vdash \phi_u \to \neg\psi$.*

Please recall that when the formula $\varphi$ contains the common knowledge operator, we take $d = 0$, so that all states are $\epsilon$-states.

**Proposition 7.** *Given $\mathsf{C}\psi \in cl_0(\varphi)$, the following are equivalent.*

1. $\vdash \phi_w \to \neg\mathsf{C}\psi$
2. *there is a state $u$ reachable from $w$ through the relation $R_r^\varphi$ such that $\vdash \phi_u \to \neg\psi$.*

The next definition specifies how to extend a $\sigma$-state for $\varphi$ to a path in the pre-model.

**Definition 8.** *Given an arbitrary $\sigma$-state $w$, we define a sequence $\delta_w$ of states and actions $w_0 \xrightarrow{d_1} w_1 \xrightarrow{d_2} \cdots$ as follows: for any $r \in N$ and $j \geq 1$,*

1. *$w_j$ is a $\sigma$-state in $W^\varphi$, and $d_j \in D$.*
2. *$w_0 = w$.*
3. *$\phi_{w_{j-1}} \wedge \bigcirc\phi_{w_j}$ is consistent.*

4. $d_j(r) = a^r_j$ *iff* $\vdash \phi_{w_{j-1}} \to does(a^r_j)$.

The following result shows such generated sequence is indeed a path of $M^\varphi$.

**Proposition 8.** *Given an arbitrary $\sigma$-state $w$, the sequence $\delta_w$ is a path of $M^\varphi$.*

*Proof.* We first show that $\delta_w$ is infinite. Suppose not that there is some state $w_l$ with no successor. By Axiom 4 and Axiom 5, such an action $a^r_j$ for each agent always exists. Then it is only the case that $\vdash \phi_{w_l} \to \neg \bigcirc \phi_s$ for all atoms $s$ of $cl_{k,r}(\varphi)$ where $k = d - |\sigma|$ if $\sigma = \tau r$, or $cl_d(\varphi)$ if $\sigma = \epsilon$. But by Proposition 3.3 and (R2) we have $\vdash \bigcirc \bigvee_{X \in \mathcal{A}_{cl_{k,r}(\varphi)}} \phi_X$ where $k = d - |\sigma|$ if $\sigma = \tau r$; $\vdash \bigcirc \bigvee_{X \in \mathcal{A}_{cl_d(\varphi)}} \phi_X$ if $\sigma = \epsilon$, which contradicts that $w_l$ is consistent. With this, it remains to show that $\delta_w$ satisfies the conditions of a path in Definition 7.

Condition 1 holds directly by the definition of $U^\varphi$ and Axiom 8. Regarding Condition 2, by Clause 3 and Axiom 8, we have $\vdash \phi_{w_{j-1}} \to legal(d_j(r))$ for any $r \in N$ and $j \geq 1$, so $d_j(r) \in L^\varphi_r(w_{j-1})$. Regarding Condition 3, assume $w_{l-1} \in T^\varphi$ and $w_{l-1} \neq w_l$ for some $l \geq 1$. Without loss of generalization, say $w_{l-1} = (\sigma, X_{l-1})$ and $w_l = (\sigma, X_l)$. Then $\vdash \phi_{X_{l-1}} \to terminal$, and there is some $\alpha \in cl_{k,r}(\varphi)$ where $k = d - |\sigma|$ if $\sigma = \tau r$, or $\alpha \in cl_d(\varphi)$ if $\sigma = \epsilon$, such that either ($\alpha \in X_{l-1}$ and $\alpha \notin X_l$) or ($\alpha \notin X_{l-1}$ and $\alpha \in X_l$). By symmetry, it suffices to show the case $\alpha \notin X_{l-1}$ and $\alpha \in X_l$. Then $\vdash \phi_{X_{l-1}} \to \neg\alpha$ and $\vdash \phi_{X_l} \to \alpha$. From the former and by Axiom 9, we get $\vdash \phi_{X_{l-1}} \to \bigcirc\neg\alpha$, so $\vdash \phi_{X_{l-1}} \to \neg \bigcirc \alpha$ by Axiom 7. While from $\vdash \phi_{X_l} \to \alpha$ and by (R2), $\vdash \bigcirc\phi_{X_l} \to \bigcirc\alpha$, which contradicts that $\phi_{X_{l-1}} \wedge \bigcirc\phi_{X_l}$ is consistent. Thus, for all $j \geq 1$, if $w_{j-1} \in T^\varphi$, then $w_{j-1} = w_j$. This completes the proof. $(\Box)$

We now come to one of the main intermediate results.

**Lemma 1.** *For every $\alpha \in cl_0(\varphi)$ and every $\epsilon$-state $w$,*

$$M^\varphi, \delta_w \models^* \alpha \textit{ iff } \vdash \phi_w \to \alpha.$$

It is routine to prove this by induction on the complexity of $\alpha$. As we noted before, the pre-model $M^\varphi$ of $\varphi$ is not an EST-model. To achieve the completeness result of EGDL, it suffices to transform the pre-model of $\phi$ into a deterministic model with an equivalent satisfiability. Inspired by [18], we redefine states as a subset of atoms and treat all the successors as a single state in the new model. The transformation is given as follows:

**Definition 9.** *Let $M^\varphi = (W^\varphi, I^\varphi, T^\varphi, \{R^\varphi_r\}_{r\in N}, \{L^\varphi_r\}_{r\in N}, U^\varphi, g^\varphi, \pi^\varphi)$ be the pre-model of $\varphi$. Then $\mathfrak{A}(M^\varphi)$ is a model $(S, I, T, \{R_r\}_{r\in N}, \{L_r\}_{r\in N}, U, g, \pi)$ based on $M^\varphi$ such that*

1. *$S$ consists of all the pairs $(\sigma, \Gamma)$ such that $\sigma$ is an index, $|\sigma| \leq d$, and*
   (a) *if $\sigma = \epsilon$ then $\Gamma$ is a non-empty subset of $\mathcal{A}_{cl_d(\varphi)}$, and*
   (b) *if $\sigma = \tau r$ then $\Gamma$ is a non-empty subset of $\mathcal{A}_{cl_{k,r}(\varphi)}$, where $k = d - |\sigma|$.*
2. *$I = \{(\sigma, \Gamma) \in S \ : \ \Gamma \subseteq \{X \ : \ (\sigma, X) \in I^\varphi\}\}$.*
3. *$T = \{(\sigma, \Gamma) \in S \ : \ \Gamma \subseteq \{X \ : \ (\sigma, X) \in T^\varphi\}\}$.*
4. *$(\sigma, \Gamma) R_r (\tau, \Delta)$ iff $\sigma \# r = \tau \# r$ and $\{\psi : \mathsf{K}_r\psi \in \bigcup \Gamma\} = \{\chi : \mathsf{K}_r\chi \in \bigcup \Delta\}$.*
5. *$L_r((\sigma, \Gamma)) = \bigcup_{X \in \Gamma} L^\varphi_r((\sigma, X))$.*
6. *$U((\sigma, \Gamma), d) = (\sigma, \Delta)$ where $\Delta = \{Y \ : \ (\sigma, Y) \in \bigcup_{X\in\Gamma} U^\varphi((\sigma, X), d)\}$.*

7. $g(r) = \{(\sigma, \Gamma) \in S \;:\; \Gamma \subseteq \{X \;:\; (\sigma, X) \in g^{\varphi}(r)\}\}$.
8. $\pi((\sigma, \Gamma)) = \bigcup_{X \in \Gamma} \pi^{\varphi}((\sigma, X))$.

The following result shows the associated model $\mathfrak{A}(M^{\varphi})$ is just what we want.

**Proposition 9.** *Given a pre-model $M^{\varphi}$ of $\varphi$, the model $\mathfrak{A}(M^{\varphi})$ is an EST-model.*

*Proof.* Clearly, $S \neq \emptyset$ and $I \cap T = \emptyset$ follows from $\vdash \neg(initial \wedge terminal)$. It is straightforward that the epistemic relation $R_r$ is equivalent. Regarding $L_r$, for any $(\sigma, \Gamma) \in S$, since $\Gamma \neq \emptyset$ and $L_r^{\varphi}((\sigma, X)) \neq \emptyset$ for any $(\sigma, X) \in W^{\varphi}$, so by definition $L_r((\sigma, \Gamma)) \neq \emptyset$ (Condition (i)). Assume $(\sigma, \Gamma) \in T$, then by the definition we have $(\sigma, X) \in T^{\varphi}$ for any $X \in \Gamma$, so $\vdash \phi_X \rightarrow terminal$. And by Axiom 3, we have $\vdash \phi_X \rightarrow \bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg legal(a^r) \wedge legal(noop^r)$, so $L_r^{\varphi}((\sigma, X)) = \{noop^r\}$ for any $X \in \Gamma$. Thus, $L_r((\sigma, \Gamma)) = \{noop^r\}$ (Condition (ii)). It remains to show that the update function $U$ satisfies the assumption.

We first show that for any state $(\sigma, \Gamma) \in S$ and joint action $d \in D$, $U((\sigma, \Gamma), d)$ is non-initial. This follows from the fact that for any $d \in D$ and $(\sigma, X) \in W^{\varphi}$, $U^{\varphi}((\sigma, X), d) \cap I^{\varphi} = \emptyset$ (by Axiom 2).

We next show that $U((\sigma, \Gamma), d)$ is unique if exists. Suppose not, then there are $(\sigma, \Delta)$ and $(\sigma, \Delta')$ such that $U((\sigma, \Gamma), d) = (\sigma, \Delta)$, $U((\sigma, \Gamma), d) = (\sigma, \Delta')$ and $\Delta \neq \Delta'$. But by the definition we have $\Delta = \Delta' = \{Y \;:\; (\sigma, Y) \in \bigcup_{X \in \Gamma} U^{\varphi}((\sigma, X), d)\}$: a contradiction. Thus, $U(s, d)$ is unique.

The last assumption follows from the fact that for any $(\sigma, X) \in T^{\varphi}$, $U^{\varphi}((\sigma, X), \langle noop^r \rangle_{r \in N}) = (\sigma, X)$. This is proved by a similar method of Proposition 8. (□)

To complete the transformation, we next show how to generate a path in $\mathfrak{A}(M^{\varphi})$ from a given path in the pre-model $M^{\varphi}$ of $\varphi$.

**Definition 10.** *Let $M^{\varphi}$ be the pre-model of $\varphi$. For any path $\delta := (\sigma, X_0) \xrightarrow{d_1} (\sigma, X_1) \xrightarrow{d_2} \cdots$ of $M^{\varphi}$, we define a sequence of states and joint actions $\widehat{\delta} := (\tau, \Gamma_0) \xrightarrow{d'_1} (\tau, \Gamma_1) \xrightarrow{d'_2} \cdots$ with respect to $\delta$ as follows: for any $j \geq 1$,*

1. *$\sigma = \tau$ and $d_j = d'_j$,*
2. *$\Gamma_0 = \{X_0\}$, and*
3. *$\Gamma_j = \{X_j : (\sigma, X_j) \in \bigcup_{X_{j-1} \in \Gamma_{j-1}} U^{\varphi}((\sigma, X_{j-1}), d_j)\}$.*

**Proposition 10.** *For any path $\delta$ of $M^{\varphi}$, the sequence $\widehat{\delta}$ is a path of $\mathfrak{A}(M^{\varphi})$.*

*Proof.* Let $\delta := (\sigma, X_0) \xrightarrow{d_1} (\sigma, X_1) \xrightarrow{d_2} \cdots$ and $\widehat{\delta} := (\tau, \Gamma_0) \xrightarrow{d_1} (\tau, \Gamma_1) \xrightarrow{d_2} \cdots$. Clearly, $\widehat{\delta}$ is infinite as $\delta$ is infinite. It suffices to show that $\widehat{\delta}$ satisfies all the conditions of a path in Definition 2. Let us first consider Condition 1. Suppose not that there is some $k \geq 1$ such that $\widehat{\delta}[k] \in I$, then by the definition of $\mathfrak{A}(M^{\varphi})$, we have for all $(\sigma, X) \in \widehat{\delta}[k]$, $(\sigma, X) \in I^{\varphi}$. In particular, $(\sigma, X_k) \in I^{\varphi}$, so $\vdash X_k \rightarrow initial$. Then by (R2) we have $\vdash \bigcirc X_k \rightarrow \bigcirc initial$. But by Axiom 2 we have $\vdash X_{k-1} \rightarrow \neg \bigcirc initial$, contradicting that the formula $X_{k-1} \wedge \bigcirc X_k$ is consistent. Thus, $\widehat{\delta}[j] \notin I$ for all $j \geq 1$. Condition 2 holds directly by the last two clauses of Definition 10. Regarding Condition 3, for any $r \in N$, we have $d_j(r) \in L_r^{\varphi}((\sigma, X_{j-1}))$ by Definition 7. Since $X_{j-1} \in \Gamma_{j-1}$, so we have $L^{\varphi}_r((\sigma, X_{j-1})) \subseteq L_r((\sigma, \Gamma_{j-1}))$ by Definition 9. Thus, $d_j(r) \in L_r((\sigma, \Gamma_{j-1}))$. Regarding Condition 4, it suffices to show the following fact that for any $(\sigma, X) \in T^{\varphi}$ and $d \in D$,

$$U^{\varphi}((\sigma, X), d) = \begin{cases} \{(\sigma, X)\} & \text{if } d = \langle noop^r \rangle_{r \in N}; \\ \emptyset & \text{otherwise.} \end{cases}$$

By Axiom 3, $\vdash \phi_X \to \bigwedge_{r \in N}(\bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg legal(a^r) \wedge legal(noop^r))$. And by Axiom 8, we have $\vdash \phi_X \to \bigwedge_{r \in N}(\bigwedge_{a^r \in A^r \setminus \{noop^r\}} \neg does(a^r))$. Thus, $U^{\varphi}((\sigma, X), d) = \emptyset$ for any $d \neq \langle noop^r \rangle_{r \in N}$. Then by Axiom 4, we have $\vdash \phi_X \to \bigwedge_{r \in N} does(noop^r)$. Since $\phi_X \wedge \bigcirc \phi_X$ is consistent, so $(\sigma, X) \in U^{\varphi}((\sigma, X), \langle noop^r \rangle_{r \in N})$. Suppose there is another state $(\tau, Y) \in W^{\varphi}$ such that $(\tau, Y) \in U^{\varphi}((\sigma, X), \langle noop^r \rangle_{r \in N})$ and $(\sigma, X) \neq (\tau, Y)$. Then there is some $\alpha \in cl_{k,r}(\varphi)$ where $k = d - |\sigma|$ if $\sigma = \tau r$, or $\alpha \in cl_d(\varphi)$ if $\sigma = \epsilon$, such that either ($\alpha \in X$ and $\alpha \notin Y$) or ($\alpha \notin X$ and $\alpha \in Y$). By symmetry, it suffices to show the case $\alpha \in X$ and $\alpha \notin Y$. Then $\vdash \phi_X \to terminal \wedge \alpha$. And by Axiom 9, we get $\vdash \phi_X \to \bigcirc \alpha$. By Proposition 3.2, we have either $\vdash \phi_Y \to \alpha$ or $\vdash \phi_Y \to \neg \alpha$. The former contradicts with the assumption $\alpha \notin Y$. Thus, $\vdash \phi_Y \to \neg \alpha$. Then by (R2), we have $\vdash \bigcirc(\phi_Y \to \neg \alpha)$, so $\vdash \bigcirc \phi_Y \to \neg \bigcirc \alpha$, contradicting that $\phi_X \wedge \bigcirc \phi_Y$ is consistent. Thus, $U^{\varphi}((\sigma, X), \langle noop^r \rangle_{r \in N}) = \{(\sigma, X)\}$. For any $(\sigma, \Gamma_{j-1}) \in T$, by the definition we have $(\sigma, X) \in T^{\varphi}$ for any $X \in \Gamma_{j-1}$, then by the fact for any $X \in \Gamma_{j-1}$, $U^{\varphi}((\sigma, X), d_j) = \{(\sigma, X)\}$ and $d_j = \langle noop^r \rangle_{r \in N}$, so $\{X_j : (\sigma, X_j) \in \bigcup_{X \in \Gamma_{j-1}} U^{\varphi}((\sigma, X), \langle noop^r \rangle_{r \in N})\} = \Gamma_{j-1}$. Thus, $\Gamma_j = \Gamma_{j-1}$. This completes the proof of the proposition. ($\Box$)

Then we have the following equivalent result in terms of the transformations.

**Lemma 2.** *Let $M^{\varphi}$ be the pre-model of $\varphi$ and $\delta$ be a path of $M^{\varphi}$. Then for any $\alpha \in \mathcal{L}$,*

$$M^{\varphi}, \delta \models^* \alpha \quad \textit{iff} \quad \mathfrak{A}(M^{\varphi}), \widehat{\delta} \models \alpha.$$

We are now in the position to prove the soundness and completeness results of EGDL with respect to the epistemic state transition models.

**Theorem 1.** *The logic EGDL is sound and complete with respect to the class of epistemic state transition models, i.e., for every $\varphi \in \mathcal{L}$, $\models \varphi$ iff $\vdash \varphi$.*

*Proof.* We only show the completeness. Assume $\nvdash \varphi$, then $\neg\varphi$ is consistent, so there is an $\epsilon$-state $w$ such that $\vdash \phi_w \to \neg\varphi$. By Lemma 1, $M^{\neg\varphi}, \delta_w \models^* \neg\varphi$. And by Lemma 2, we have $\mathfrak{A}(M^{\neg\varphi}), \widehat{\delta_w} \models \neg\varphi$. Thus, $\not\models \varphi$. This completes the proof. ($\Box$)

Then we have the following result saying that EGDL has the *finite model property*.

**Theorem 2.** *Let $\varphi$ be a formula in EGDL. If $\varphi$ is satisfiable, then it is satisfiable in a finite epistemic state transition model.*

## 4 Related Work

To deal with imperfect information games, many logics, mostly epistemic extensions of Alternating-time Temporal Logic, Strategy Logic and PDL, have been developed [1, 2, 10–13]. Different from them, as shown in [14], EGDL uses a bottom-up approach in order to create a balance between expressive power and computational efficiency. It is a conservative extension of a simple and practical logical language GDL. Besides the literature discussed in Introduction, the following is also worth mentioning.

Zhang and Thielscher propose a dynamic extension of GDL for reasoning about game strategies, and develop a sound and complete axiomatic system for this logic [24]. With different languages and semantics, their axiomatization and techniques to prove the completeness are different from ours. In particular, they make use of forgetting techniques while we combine techniques used for completeness of PDL and ETLs.

As a logic programming language, GDL has recently been extended to GDL-II and GDL-III so as to incorporate imperfect information games [22, 23]. They are different from EGDL in two aspects: (i) GDL-II and GDL-III are purely logic programming languages and do not provide a reasoning facility to reason about epistemic game rules. While as a logic EGDL is able to represent and reason about rules of imperfect information. Moreover, we have developed an axiomatic system for EGDL. (ii) GDL-II and GDL-III considers games with perfect recall players and randomness, such as dice rolling and card shuffling. While EGDL focuses on imperfect recall games without randomness. Yet EGDL is flexible enough to specify perfect recall as well as the state-based memory and the action-based memory [4].

Finally, it is worth mentioning that EGDL has similarities with ETLs such as CK-$\mathrm{L}_m$ [7], but they are significantly different in the following ways: (i) With $does(.)$ operator, EGDL can express actions and their effects, thus it can be used for reasoning about actions, while epistemic temporal logics are not. Moreover, with action operator, the completeness proof of EGDL is different from those of epistemic temporal logics; (ii) EGDL contains a single temporal operator ("next"), and can only represent finite steps of time. (iii) Model checking for EGDL is in $\Delta_2^p$, while, for epistemic temporal logics, it is at least PSPACE-hard [21].

## 5 Conclusion

We have developed a sound and complete axiomatic system for a variant of EGDL. From the completeness proof, we have derived the finite model property of this logic. We have also demonstrated how to use the proof theory for inferring game properties from game rules.

Directions of future research are manifold. We intend to investigate the satisfiability problem of EGDL. The hardness of the satisfiability problem for EGDL follows from the fact that EGDL is a conservative extension of $S5_n^C$, and the satisfiability problem for $S5_n^C$ is EXPTIME-complete [8]. We also want to study the definability problem of EGDL [3]: which properties of games are definable by means of EGDL-formulas? For instance, this paper shows that EGDL is able to provide a description for Krieg-Tictactoe. It would be interesting to consider the other direction: whether Krieg-Tictactoe is completely or even uniquely specified by such a description.

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## References


1. van Benthem, J.: Games in dynamic-epistemic logic. Bulletin of Economic Research 53(4), 219–248 (2001)
2. van Benthem, J.: Logic in games. MIT Press (2014)
3. Blackburn, P., De Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press (2002)
4. van Ditmarsch, H., Knight, S.: Partial information and uniform strategies. In: CLIMA'14. pp. 183–198 (2014)
5. Fagin, R., Moses, Y., Halpern, J.Y., Vardi, M.Y.: Reasoning about Knowledge. MIT press (2003)
6. Genesereth, M., Love, N., Pell, B.: General game playing: Overview of the AAAI competition. AI magazine 26(2), 62–72 (2005)
7. Halpern, J.Y., van Der Meyden, R., Vardi, M.Y.: Complete axiomatizations for reasoning about knowledge and time. SIAM Journal on Computing 33(3), 674–703 (2004)
8. Halpern, J.Y., Moses, Y.: A guide to completeness and complexity for modal logics of knowledge and belief. Artificial intelligence 54(3), 319–379 (1992)
9. Hintikka, J.: Knowledge and Belief: An Introduction to the Logic of the Two Notions. Cornell University Press (1962)
10. van der Hoek, W., Pauly, M.: Modal logic for games and information. In: Blackburn, P., van Benthem, J., Wolter, F. (eds.) Handbook of Modal Logic, vol. 3, pp. 1077–1148. Elsevier (2006)
11. van der Hoek, W., Wooldridge, M.: Cooperation, knowledge, and time: Alternating-time temporal epistemic logic and its applications. Studia Logica 75(1), 125–157 (2003)
12. Huang, X., van der Meyden, R.: An epistemic strategy logic. In: SR'14. pp. 35–41 (2014)
13. Jamroga, W., van der Hoek, W.: Agents that know how to play. Fundamenta Informaticae 63(2), 185–219 (2004)
14. Jiang, G., Zhang, D., Perrussel, L., Zhang, H.: Epistemic GDL: A logic for representing and reasoning about imperfect information games. In: IJCAI'16. pp. 1138–1144 (2016)
15. Kozen, D., Parikh, R.: An elementary proof of the completeness of PDL. Theoretical Computer Science 14(1), 113–118 (1981)
16. Love, N., Hinrichs, T., Haley, D., Schkufza, E., Genesereth, M.: General game playing: Game description language specification. Stanford Logic Group. `http://logic.stanford.edu/reports/LG-2006-01.pdf` (2006)
17. Pritchard, D.B.: The Encyclopedia of Chess Variants. Games & Puzzles (1994)
18. Rabin, M.O., Scott, D.: Finite automata and their decision problems. IBM journal of research and development 3(2), 114–125 (1959)
19. Schiffel, S., Thielscher, M.: Reasoning about general games described in GDL-II. In: AAAI'11. pp. 846–851 (2011)
20. Schobbens, P.Y.: Alternating-time logic with imperfect recall. Electronic Notes in Theoretical Computer Science 85(2), 82–93 (2004)
21. Sistla, A.P., Clarke, E.M.: The complexity of propositional linear temporal logics. Journal of the ACM 32(3), 733–749 (1985)
22. Thielscher, M.: A general game description language for incomplete information games. In: AAAI'10. pp. 994–999 (2010)
23. Thielscher, M.: GDL-III: A proposal to extend the game description language to general epistemic games. In: ECAI'16. pp. 1630–1631 (2016)
24. Zhang, D., Thielscher, M.: A logic for reasoning about game strategies. In: AAAI'15. pp. 1671–1677 (2015)
25. Zhang, D., Thielscher, M.: Representing and reasoning about game strategies. Journal of Philosophical Logic 44(2), 203–236 (2015)